

FUNDAMENTAL GROUPS OF NEIGHBORHOOD COMPLEXES

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ABSTRACT. We introduce the notions of 2-covering maps and 2-fundamental groups of graphs, and investigate their basic properties. These concepts are closely related to Hom complexes and neighborhood complexes. Indeed, we prove that the fundamental group of a neighborhood complex is isomorphic to a subgroup of the 2-fundamental group whose index is 1 or 2. We prove that the 2-fundamental group and the fundamental group of a neighborhood complex for a connected graph whose chromatic number is 3 have group homomorphisms onto \mathbb{Z} .

1. INTRODUCTION

Neighborhood complexes were defined by Lovász in [10] in the context of the graph coloring problem. He proved the connectivity of the neighborhood complex gives the lower bound of the chromatic number and determined the chromatic number of Kneser's graphs. In this paper, we give another interpretation of the fundamental group of the neighborhood complex, and obtain a necessary condition for a graph that its chromatic number is 3. Usually, a relation between the chromatic number and the topology of the graph complex is obtained by using characteristic classes of principal bundles of finite groups (see [7], [9], and [12]), but our method is different.

We introduce the notions of 2-covering maps and 2-fundamental groups of graphs. 2-covering maps are essentially different from usual covering maps of graphs. In fact, we prove that the connected 2-covering over a complete graph K_n on n vertices for $n \geq 4$ are only K_n and $K_2 \times K_n$ (Corollary 5.2). We study that 2-fundamental groups are closely related to 2-covering maps, as is the case of the covering space theory in topology. For example, there is a natural correspondence between subgroups of the 2-fundamental group of a graph G and the connected based 2-coverings over G (Theorem 4.16). After establishing the basic theory of 2-covering maps and 2-fundamental groups, we prove that a 2-fundamental group of a connected graph whose chromatic number is 3 has a surjection to \mathbb{Z} (Corollary 5.4). Finally, we prove that the subgroup of a 2-fundamental group, called the even part, whose index is 1 or 2 is isomorphic to the fundamental group of the neighborhood complex (Theorem 6.1). Then we prove that the 1-dimensional homology group of $\mathcal{N}(G)$ has \mathbb{Z} as a direct summand (Corollary 6.2).

The rest of this paper organized as follows. In Section 2 we review the all necessary definitions and facts related to graphs, simplicial complexes, neighborhood complexes, and Hom complexes. In Section 3 we provide the definition of 2-covering maps and study the basic properties of 2-covering maps. Here we prove that 2-covering maps induce covering maps of neighborhood complexes and Hom complexes. In Section 4 we provide the definition of 2-fundamental groups and study their basic properties and investigate the relation between 2-fundamental groups and based 2-covering maps. In Section 5, we compute 2-fundamental groups for basic graphs and establish van Kampen theorem for 2-fundamental

groups. In Section 6, we prove the even part of the 2-fundamental group of G is isomorphic to the fundamental group of the neighborhood complex of G .

2. DEFINITIONS AND FACTS

In this section, we review the basic definitions and facts used in this paper relating graphs following [1], [7] and [8]. For the covering space theory in topology, we refer to [5].

Graphs : A *graph* is a pair (V, E) where V is a set and E is a subset of $V \times V$ such that $(x, y) \in E$ implies $(y, x) \in E$. Therefore our graphs are undirected, simple and may have loops. For a graph $G = (V, E)$, V is called the *vertex set* of G and E is called the *edge set* of G . We write $V(G)$ for the vertex set of G and $E(G)$ for the edge set of G . For vertices v, w of G , we write $v \sim w$ if $(v, w) \in E(G)$.

Let G and H be graphs. A map $f : V(G) \rightarrow V(H)$ is called a *graph homomorphism* or a *graph map* from G to H if $f \times f(E(G)) \subset E(H)$.

Let G be a graph. A subset A of $V(G)$ is called independent if $A \times A \cap E(G) = \emptyset$.

We define the graph K_n for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ by $V(K_n) = \{0, 1, \dots, n-1\}$ and $E(K_n) = \{(x, y) \in V(G) \times V(G) \mid x \neq y\}$, which is called the complete graph on n -vertices. For a graph G , a graph homomorphism from G to K_n is called an *n -coloring* of G . Set

$$\chi(G) = \min\{n \mid \text{There is a graph homomorphism from } G \text{ to } K_n.\}.$$

In this paper, we set $\chi(G) = \infty$ if there is no n such that there is a graph homomorphisms from G to K_n . $\chi(G)$ is called the *chromatic number* of G . We say that G is *bipartite* if $\chi(G) = 2$.

A graph G is said to be *connected* if $V(G) \neq \emptyset$ and each pair $(v, w) \in V(G) \times V(G)$ there exists a finite sequence (v_0, \dots, v_n) of vertices of G such that $v_0 = v$, $v_n = w$, and $(v_{i-1}, v_i) \in E(G)$ for every $i \in \{1, \dots, n\}$.

Let G be a graph. A graph H is called a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Let $\{H_\alpha\}_{\alpha \in A}$ be a family of subgraphs of G . The subgraph $(\bigcup_{\alpha \in A} V(H_\alpha), \bigcup_{\alpha \in A} E(H_\alpha))$ of G is called the *union* of $\{H_\alpha\}_{\alpha \in A}$, written by $\bigcup_{\alpha \in A} H_\alpha$, and the subgraph $(\bigcap_{\alpha \in A} V(H_\alpha), \bigcap_{\alpha \in A} E(H_\alpha))$ of G is called the *intersection* of $\{H_\alpha\}_{\alpha \in A}$, and written by $\bigcap_{\alpha \in A} H_\alpha$.

Let $\{G_\alpha\}_{\alpha \in A}$ be a family of graphs. We define the *product* $\prod_{\alpha \in A} G_\alpha$ of $\{G_\alpha\}_{\alpha \in A}$ by setting $V(\prod_{\alpha \in A} G_\alpha) = \prod_{\alpha \in A} V(G_\alpha)$ and $E(\prod_{\alpha \in A} G_\alpha) = \{((x_\alpha)_{\alpha \in A}, (y_\alpha)_{\alpha \in A}) \mid (x_\alpha, y_\alpha) \in E(G_\alpha) \text{ for each } \alpha \in A.\}$. We define the *coproduct* $\coprod_{\alpha \in A} G_\alpha$ of $\{G_\alpha\}_{\alpha \in A}$ by setting $V(\coprod_{\alpha \in A} G_\alpha) = \coprod_{\alpha \in A} V(G_\alpha)$ and $E(\coprod_{\alpha \in A} G_\alpha) = \coprod_{\alpha \in A} E(G_\alpha)$.

A right action of a group Γ on a graph G is a right action $\alpha : V(G) \times \Gamma \rightarrow V(G)$ of Γ on $V(G)$ as a set such that $\alpha(x, \gamma) \sim \alpha(y, \gamma)$ if $x \sim y$ for all $x, y \in V(G)$ and $\gamma \in \Gamma$. A graph G with a right action of Γ on G is called a *right Γ -graph*. We define similarly a left Γ -graph.

Let G be a graph and R an equivalence relation on $V(G)$. We define the graph G/R by setting

$$V(G/R) = V(G)/R$$

$$E(G/R) = \{(\alpha, \beta) \mid \alpha \times \beta \cap E(G) \neq \emptyset\}.$$

Then the quotient map $q : V(G) \rightarrow V(G/R)$ is a graph homomorphism. For a graph homomorphism $f : G \rightarrow H$ such that $f(v) = f(w)$ for all $(v, w) \in R$, we can easily see that there exists a unique graph homomorphism $\bar{f} : G/R \rightarrow H$ such that $\bar{f} \circ q = f$.

Let Γ be a group and G a right Γ -graph. We define the equivalence relation R_Γ on $V(G)$ by setting

$$R_\Gamma = \{(v, v\gamma) \mid v \in V(G), \gamma \in \Gamma\}.$$

We write G/Γ for G/R_Γ . We define similarly $\Gamma \backslash H$ for a left Γ -graph H .

Let G be a graph and $v \in V(G)$. We write $N(v)$ for the set $\{w \in V(G) \mid (v, w) \in E(G)\}$. $N(v)$ is called the *neighborhood* of v in G . We say that v is *isolated* if $N(v) = \emptyset$. In general, for a subset A of $V(G)$, we write $N(A)$ for the set $\{w \in V(G) \mid \text{There is } u \in A \text{ such that } (u, w) \in E(G)\}$. We write $N_2(v)$ for $N(N(v))$.

A *based graph* is a pair (G, v) where G is a graph and v is a vertex of G .

Let (G, v) and (H, w) be based graphs. A graph homomorphism f from G to H such that $f(v) = w$ is called a *based graph homomorphism* or a *based graph map* from (G, v) to (H, w) .

Simplicial complex : A pair (V, Δ) is called an *abstract simplicial complex* or a *simplicial complex* if (V, Δ) satisfies the following properties.

- (0) V is a set and Δ is a subset of 2^V .
- (1) Each $\sigma \in \Delta$ is a finite subset of V .
- (2) For each $v \in V$, $\{v\} \in \Delta$.
- (3) Let $\tau, \sigma \in 2^V$ with $\tau \subset \sigma$. If $\sigma \in \Delta$, then $\tau \in \Delta$.

Let (V, Δ) be a simplicial complex. V is called a *vertex set* of (V, Δ) . We often abbreviate (V, Δ) to Δ for a simplicial complex (V, Δ) . In this notation, the vertex set of Δ is written by $V(\Delta)$.

Let Δ and Δ' be simplicial complexes. We say that Δ' is a *subcomplex* of Δ if $V(\Delta') \subset V(\Delta)$ and $\Delta' \subset \Delta$, and written by $\Delta' \subset \Delta$.

Let Δ be a simplicial complex and $v \in V(\Delta)$. We define the *star* of v , written by $\text{st}(v)$, as the subcomplex $\{\sigma \in \Delta \mid \sigma \cup \{v\} \in \Delta\}$ of Δ .

Let Δ_1 and Δ_2 be simplicial complexes. A map $f : V(\Delta_1) \rightarrow V(\Delta_2)$ is called a *simplicial map* if $f(\sigma) \in \Delta_2$ for each $\sigma \in \Delta_1$.

A partially ordered set is called a *poset*. Let P be a poset. A subset P' of P is called a *chain* of P if the restriction of the partial order of P to P' is a total order of P' . We set $\Delta(P) = \{P' \mid P' \text{ is a finite chain of } P\}$. Then $\Delta(P)$ forms a simplicial complex and is called the *order complex* of P . Let $f : P \rightarrow Q$ be an order preserving map. Since f preserves finite chains, we have a simplicial map $\Delta(f) : \Delta(P) \rightarrow \Delta(Q)$.

Let V be a set. We write $\mathbb{R}^{(V)}$ for a free \mathbb{R} -module generated by V . We regard $\mathbb{R}^{(V)}$ as a topological space with the direct limit topology of finite dimensional vector subspaces of $\mathbb{R}^{(V)}$. For a finite subset $S \subset V$, the topological subspace $\{\sum_{i=0}^n a_i v_i \mid v_i \in S, a_i \geq 0, \sum_{i=1}^n a_i = 1\}$ of $\mathbb{R}^{(V)}$ is written by Δ_S .

Let Δ be a simplicial complex. The topological subspace

$$|\Delta| = \bigcup_{\sigma \in \Delta} \Delta_\sigma$$

of $\mathbb{R}^{(V(\Delta))}$ is called the *geometrical realization* of Δ .

For a poset P , the geometrical realization of $\Delta(P)$ is called the geometrical realization of P , and we write $|P|$ for $|\Delta(P)|$.

Hom complex : Let G and H be graphs. A *multihomomorphism* from G to H is a map $\eta : V(G) \rightarrow 2^{V(H)}$ such that $\eta(v) \times \eta(w) \subset E(H)$ for all $(v, w) \in E(G)$. For multihomomorphisms η, η' from G to H , we write $\eta \leq \eta'$ if $\eta(v) \subset \eta'(v)$ for each $v \in V(G)$. Then \leq is a partial order of the set of all multihomomorphisms from G to H and we write $\text{Hom}(G, H)$ for this poset. $\text{Hom}(G, H)$ is called the *Hom complex* from G to H . We remark that a minimal point of $\text{Hom}(G, H)$ is identified with a graph homomorphism from G to H .

Let G, H_1 and H_2 be graphs and $f : H_1 \rightarrow H_2$ a graph homomorphism. Then we have the poset map $f_* : \text{Hom}(G, H_1) \rightarrow \text{Hom}(G, H_2)$ by setting $(f_*\eta)(x) = f(\eta(x))$ for $x \in V(G)$ and $\eta \in \text{Hom}(G, H_1)$. And we have the poset map $f^* : \text{Hom}(H_2, G) \rightarrow \text{Hom}(H_1, G)$ by setting $f^*(\eta)(x) = \eta(f(x))$ for $x \in V(H_1)$ and $\eta \in \text{Hom}(H_2, G)$.

In some literature, Hom complex is defined as follows for finite graphs. Let G and H be finite graphs. We remark that there is a natural correspondence between the set of the cells of $|\Delta^{V(H)}|$ and the set of subsets of $V(H)$. In this correspondence, a multihomomorphism from G to H determines a cell of $\prod_{v \in V(G)} |\Delta^{V(H)}|$. We write $X_{G,H}$ for the union of all cells of $\prod_{v \in V(G)} |\Delta^{V(H)}|$ determined by the multihomomorphisms from G to H . We remark that the definition of $X_{G,H}$ needs the assumption that G and H are finite graphs. Some people say that $X_{G,H}$ is the Hom complex from G to H . However, since $X_{G,H}$ is a regular CW-complex and the face poset of $X_{G,H}$ is identified with $\text{Hom}(G, H)$ we defined, $X_{G,H}$ is naturally homeomorphic to $|\text{Hom}(G, H)|$. But since we want to consider the infinite graphs, we defined Hom complex as above.

For more details and interests about Hom complexes, see [1], [7], [8], [13].

\times -homotopy theory : Dochtermann established a homotopy theory of graphs in [3] called \times -homotopy theory as follows.

For $n \in \mathbb{N} = \{0, 1, 2, \dots\}$, we define the graph I_n by $V(I_n) = \{0, 1, \dots, n\}$ and $E(I_n) = \{(x, y) \mid |x - y| \leq 1\}$. Let $f, g : G \rightarrow H$ be graph homomorphisms. A \times -homotopy from f to g is a graph homomorphism $F : G \times I_n \rightarrow H$ for some $n \in \mathbb{N}$ such that $F(x, 0) = f(x)$ and $F(x, n) = g(x)$ for each $x \in V(G)$. If there exists a \times -homotopy from f to g , f is said to be \times -homotopic to g and written by $f \simeq_\times g$ or simply by $f \simeq g$.

We can see that f is \times -homotopic to g if and only if f and g are in the same connected component of $\text{Hom}(G, H)$. In fact, for graph homomorphisms f, g from G to H , the set map $F : V(G \times I_1) \rightarrow V(H)$ defined by $(x, 0) \mapsto f(x)$ and $(x, 1) \mapsto g(x)$ is a graph homomorphism if and only if the set map $V(G) \rightarrow 2^{V(H)} - \{\emptyset\}$ defined by $x \mapsto \{f(x), g(x)\}$ is a multihomomorphism.

A graph homomorphism $f : G \rightarrow H$ is called a \times -homotopy equivalence if there exists a graph homomorphism $g : H \rightarrow G$ such that $gf \simeq_\times \text{id}_G$ and $fg \simeq_\times \text{id}_H$.

An important example of \times -homotopy equivalences is a folding homomorphism defined as follows. Given a graph G and $v \in V(G)$, we write $G \setminus v$ for the graph $V(G \setminus v) = V(G) \setminus \{v\}$ and $E(G \setminus v) = E(G) \cap V(G \setminus v) \times V(G \setminus v)$. The graph $G \setminus v$ is called a *fold* of G if there exists $w \in V(G)$ such that $N(v) \subset N(w)$. In this case $f_v : V(G) \rightarrow V(G \setminus v)$ defined by $x \mapsto x$ for $x \neq v$ and $v \mapsto w$ is called a

folding map. It is easy to see that f_v is a \times -homotopy equivalence and its \times -homotopy inverse is the inclusion $G \setminus v \rightarrow G$.

Neighborhood complex : For a graph G , a neighborhood complex is the simplicial complex

$$\mathcal{N}(G) = \{A \subset V(G) \mid \#A < \infty \text{ and there exists } v \in V(G) \text{ such that } A \subset N(v)\}.$$

We remark that a vertex of $\mathcal{N}(G)$ is a non-isolated vertex of G . If G is finite, $\mathcal{N}(G)$ is known to be homotopy equivalent to $\text{Hom}(K_2, G)$ (see [1]). Moreover, $\mathcal{N}(G)$ is known to be simple homotopy equivalent to $\text{Hom}(K_2, G)$ (see [6]). For more details and interests about neighborhood complexes, see [2], [10], [14].

3. 2-COVERING MAPS OF GRAPHS

In this section, we introduce the notion of a 2-covering map and investigate its basic properties.

Definition of 2-covering maps :

Given a graph G and a vertex v of G . Recall that the set $\{w \in V(G) \mid (v, w) \in E(G)\}$ is written by $N(v)$ and $\{w \in V(G) \mid \text{There is } u \in V(G) \text{ such that } (v, u), (u, w) \in E(G)\}$ is written by $N_2(v)$.

Definition 3.1. A graph homomorphism $p : G \rightarrow H$ is called a *2-covering map* if $p|_{N(v)} : N(v) \rightarrow N(pv)$ and $p|_{N_2(v)} : N_2(v) \rightarrow N_2(pv)$ are both bijective for each $v \in V(G)$. A 2-covering map $p : G \rightarrow H$ is said to be *connected* if G is connected.

We do not assume that a 2-covering map is surjective. Thus the inclusion $\emptyset \rightarrow G$ is a 2-covering map.

Example 3.2. (1) An identity map is a 2-covering map.

(2) For a graph G , the second projection $K_2 \times G \rightarrow G$ is a 2-covering map. We remark that $K_2 \times G$ is 2-colorable for any G . Suppose G is connected. Then $K_2 \times G$ is connected if and only if $\chi(G) \geq 3$. In Section 5, we prove that the only connected 2-coverings over K_n for $n \geq 4$ are K_n and $K_2 \times K_n$.

(3) The cyclic graph C_n for $n \geq 3$ is defined by $V(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $E(C_n) = \{(x, x+1), (x+1, x) \mid x \in \mathbb{Z}/n\mathbb{Z}\}$. If $n \neq 4$, then the graph homomorphism $p : C_{nk} \rightarrow C_n$ defined by $p(x \bmod nk) = p(x \bmod n)$ is a 2-covering map. Since $K_3 = C_3$, there exist infinitely many connected 2-coverings over K_3 . We remark that these graph homomorphisms do not preserve $N_3(x)$ bijectively for each $x \in C_{nk}$, where $N_3(x) = N(N_2(x))$, except for $C_6 \cong K_2 \times K_3 \rightarrow K_3$.

(4) The graph L is defined by $V(L) = \mathbb{Z}$ and $E(L) = \{(x, y) \mid |x - y| = 1\}$. If $n \geq 3$ and $n \neq 4$, the graph homomorphism p from L to C_n defined by $p(x) = (x \bmod n)$ is a 2-covering map.

Lemma 3.3. Let p be a graph homomorphism from a graph G to a graph H . If $p|_{N(v)} : N(v) \rightarrow N(pv)$ is surjective and $p|_{N_2(v)} : N_2(v) \rightarrow N_2(pv)$ is injective for each $v \in V(G)$, then p is a 2-covering map.

Proof. Let $v \in V(G)$. We want to show that $p|_{N(v)}$ is injective and $p|_{N_2(v)}$ is surjective. Let $w_1, w_2 \in N(v)$ with $p(w_1) = p(w_2)$. Since $w_1, w_2 \in N_2(w_1)$, we have $w_1 = w_2$ from the injectivity of $p|_{N_2(w_1)}$. Therefore $p|_{N(v)}$ is injective. Let $x \in N_2(pv)$. Then there exists $y \in N(pv)$ such that $x \in N(y)$. We have $w \in N(v)$ with $p(w) = y$ from the surjectivity of $p|_{N(v)}$, and $u \in N(w)$ with $p(u) = x$ from the surjectivity of $p|_{N(w)}$. Since $u \in N_2(v)$, we have $p|_{N_2(v)}$ is surjective. \square

Lemma 3.4. *Let $f : G \rightarrow H$, $g : H \rightarrow K$ be graph homomorphisms. Then the followings hold.*

- (1) *If g and f are 2-covering maps, then gf is a 2-covering map.*
- (2) *If g and gf are 2-covering maps, then f is a 2-covering map.*
- (3) *If f is a surjective 2-covering map and gf is a 2-covering map, then g is a 2-covering map.*

Proof. Let $v \in V(G)$. We have the following commutative diagrams.

$$\begin{array}{ccc}
 N(v) & \xrightarrow{f|_{N(v)}} & N(f(v)) \\
 gf|_{N(v)} \downarrow & & \downarrow g|_{N(f(v))} \\
 N(gf(v)) & \xlongequal{\quad} & N(gf(v))
 \end{array}
 \qquad
 \begin{array}{ccc}
 N_2(v) & \xrightarrow{f|_{N_2(v)}} & N_2(f(v)) \\
 gf|_{N_2(v)} \downarrow & & \downarrow g|_{N_2(f(v))} \\
 N_2(gf(v)) & \xlongequal{\quad} & N_2(gf(v))
 \end{array}$$

If one of (1), (2), (3) holds, two of three arrows in each diagram are bijective. Hence so is third. \square

Proposition 3.5. *Let G and H be graphs and $p : G \rightarrow H$ be a 2-covering map. Then the map $|\mathcal{N}(G)| \rightarrow |\mathcal{N}(H)|$ induced by p is a covering map in a topological sense.*

Proof. Firstly, we remark that a vertex of a neighborhood complex is a non-isolated vertex.

It is sufficient to prove that $p^{-1}(\text{st}(v)) = \coprod_{v_i \in p^{-1}(v)} \text{st}(v_i)$ for each $v \in V(H)$ with $N(v) \neq \emptyset$, where $p^{-1}(\text{st}(v))$ is the subcomplex of $\mathcal{N}(G)$ whose simplex is $\sigma \in \mathcal{N}(G)$ such that $p(\sigma) \in \text{st}(v)$.

Suppose $w \in V(\text{st}(v_i)) \cap V(\text{st}(v_j))$ for $v_i, v_j \in p^{-1}(v)$. Since $v_i, v_j \in N_2(w)$ and $p(v_i) = p(v_j)$, we have $v_i = v_j$. Therefore $\text{st}(v_i)$ for $v_i \in p^{-1}(v)$ are disjoint.

Let $\sigma \in \mathcal{N}(G)$. Suppose $\emptyset \neq \sigma \in p^{-1}(\text{st}(v))$. Then there exists $v' \in V(G)$ such that $p(\sigma) \cup \{v\} \subset N(v')$. Since p is a 2-covering map, there exists $w' \in p^{-1}(v')$ such that $\sigma \subset N(w')$. Let $w \in N(w')$ with $p(w) = v$. Then we have $\sigma \in \text{st}(w)$ with $w \in p^{-1}(v)$. Therefore we have $p^{-1}(\text{st}(v)) \subset \coprod_{v_i \in p^{-1}(v)} \text{st}(v_i)$. On the other hand, $\coprod_{v_i \in p^{-1}(v)} \text{st}(v_i) \subset p^{-1}(\text{st}(v))$ is obvious. \square

Proposition 3.6. *Let T be a connected graph having no isolated points and $p : G \rightarrow H$ be a 2-covering map and $\eta_0, \eta_1 \in \text{Hom}(T, G)$ with $p_*\eta_0 = p_*\eta_1$. If there exists $x \in V(T)$ such that $\eta_0(x) \cap \eta_1(x) \neq \emptyset$, then $\eta_0 = \eta_1$.*

Proof. Let $v \in \eta_0(x) \cap \eta_1(x)$. Since T has no isolated points, we have $\eta_0(x) \subset N_2(v)$ and $\eta_1(x) \subset N_2(v)$. Since $p(\eta_0(x)) = p(\eta_1(x))$ and $p|_{N_2(v)}$ is injective, we have $\eta_0(x) = \eta_1(x)$.

Since T is connected and $\eta_0(x) = \eta_1(x)$, it is sufficient to show that, for each $(y, z) \in E(G)$, $\eta_0(y) = \eta_1(y)$ implies $\eta_0(z) = \eta_1(z)$. Thus let $(y, z) \in E(G)$ and suppose $\eta_0(y) = \eta_1(y)$ and let $w \in \eta_0(y)$. Since $\eta_0(z) \subset N(w)$ and $\eta_1(z) \subset N(w)$, $\eta_0(z) = \eta_1(z)$ from the injectivity of $p|_{N(w)}$. \square

Proposition 3.7. *Let T be a graph having no isolated points and $p : G \rightarrow H$ be a 2-covering map. Let $\eta \in \text{Hom}(T, G)$ and put $\zeta = p_*\eta$.*

- (1) *For each $\zeta_0 \leq \zeta$, there exists a unique $\eta_0 \in \text{Hom}(T, G)$ such that $p_*\eta_0 = \zeta_0$ and $\eta_0 \leq \eta$.*
- (2) *For each $\zeta_1 \geq \zeta$, there exists a unique $\eta_1 \in \text{Hom}(T, G)$ such that $p_*\eta_1 = \zeta_1$ and $\eta_1 \geq \eta$.*

Proof. Firstly, we prove (2). Choose $v_x \in \eta_0(x)$ for each $x \in V(T)$. Define a map $\eta_1 : V(T) \rightarrow 2^{V(G)} - \{\emptyset\}$ by $\eta_1(x) = (p|_{N_2(v_x)})^{-1}(\zeta_1(x))$. Since T has no isolated points, $\emptyset \neq \zeta_1(x) \subset N_2(p(v_x))$. Hence $\eta_1(x)$ is not empty. We must prove η_1 is a multihomomorphism. Let $(x, y) \in E(T)$, and $v \in \eta_1(x)$ and $w \in \eta_1(y)$. We want to show $(v, w) \in E(G)$. Since $p(v) \in \zeta_1(x) \subset N(p(v_y))$, there exists $v' \in N(v_y)$ such that

$p(v) = p(v')$. Since $v, v' \in N_2(v_x)$, we have $v = v'$ from the injectivity of $p|_{N_2(v_x)}$. Therefore we have $v \sim v_y$. Similarly we have $v_x \sim w$. So we have

$$v \sim v_y \sim v_x \sim w$$

and $p(v) \sim p(w)$. There exists $w' \in N(v)$ such that $p(w') = p(w)$. Since $w, w' \in N_2(v_y)$, we have $w = w'$. Therefore we have $v \sim w$ and η_1 is a multihomomorphism.

We prove the uniqueness of η_1 . Let $\eta' \in \text{Hom}(T, G)$ such that $\eta \leq \eta'$ and $p_*(\eta') = \zeta_1$. Let $x \in V(T)$. Since $\eta_1(x)$ and $\eta'(x)$ are subset of $N_2(v_x)$, and $p(\eta_1(x)) = p(\eta'(x))$, we have $\eta_1(x) = \eta'(x)$.

The proof of (1) is similar and much easier. Indeed, we construct η_0 by setting $\eta_0(x) = (p|_{N_2(v_x)})^{-1}(\zeta_0(x))$. η_0 is obviously a multihomomorphism since $\eta_0(x) \subset \eta(x)$. \square

Corollary 3.8. (*Homotopy Lifting Property*) *Let G, H and T be graphs and $p : G \rightarrow H$ be a 2-covering map. Suppose T has no isolated vertices. Given graph homomorphisms $F : T \times I_n \rightarrow G$ and $f : T \rightarrow H$ such that $F(x, 0) = pf(x)$ for each $x \in V(T)$. Then there exists a unique graph homomorphism $\tilde{F} : T \times I_n \rightarrow G$ such that $\tilde{F}(x, 0) = f(x)$ for each $x \in V(T)$ and $p\tilde{F} = F$.*

Proof. We can assume $n = 1$. The map $\eta : V(T) \rightarrow 2^{V(H)}, x \mapsto \{F(x, 0), F(x, 1)\}$ forms a multihomomorphism. From Proposition 3.7, we have a multihomomorphism $\tilde{\eta} : V(T) \rightarrow 2^{V(G)}$ such that $f \leq \tilde{\eta}$, and a graph homomorphism $g : T \rightarrow G$ such that $g \leq \tilde{\eta}$ and $pg(x) = F(x, 1)$. We define a set map $\tilde{F} : V(T \times I_1) \rightarrow V(G)$ by $\tilde{F}(x, 0) = f(x)$ and $\tilde{F}(x, 1) = g(x)$. Then \tilde{F} is a graph homomorphism since $f, g \leq \tilde{\eta}$. The uniqueness of \tilde{F} is obvious since $\tilde{F}(x, 1)$ is the unique element of $N_2(f(x))$ mapped to $F(x, 1)$ by p . \square

Corollary 3.9. *Let T be a graph having no isolated points and $p : G \rightarrow H$ be a 2-covering map. Then the map $|p_*| : |\text{Hom}(T, G)| \rightarrow |\text{Hom}(T, H)|$ is a covering map in a topological sense.*

Proof. This is obtained from Proposition 3.7 and the following lemma. \square

Lemma 3.10. *Let P, Q be posets and $f : P \rightarrow Q$ be an order preserving map. If the following two conditions are satisfied, then $|f| : |P| \rightarrow |Q|$ is a covering map.*

- (1) *For $x \in P$ and $y \in Q$ with $y \leq p(x)$, there exists a unique $x' \in P$ such that $p(x') = y$ and $x' \leq x$.*
- (2) *For $x \in P$ and $y \in Q$ with $y \geq p(x)$, there exists a unique $x' \in P$ such that $p(x') = y$ and $x' \geq x$.*

Proof. Firstly, we remark that, for an arbitrary poset P and $x, y \in P$, $y \in V(\text{st}(x))$ if and only if y is comparable with x .

Let $f : P \rightarrow Q$ be an order preserving map satisfying (1) and (2). It is sufficient to show

$$p^{-1}(\text{st}(y)) = \coprod_{y' \in p^{-1}(y)} \text{st}(y')$$

for each $y \in Q$. Firstly, we prove that the union of the right of the above equation is disjoint. Let $y_1, y_2 \in p^{-1}(y)$ with $y_1 \neq y_2$. Suppose there exists $x \in V(\text{st}(y_1)) \cap V(\text{st}(y_2))$. Then one of the following conditions holds.

- (i) $x \leq y_1$ and $x \leq y_2$.
- (ii) $y_1 \leq x \leq y_2$.
- (iii) $y_1 \leq x$ and $y_2 \leq x$.
- (iv) $y_2 \leq x \leq y_1$.

(i) contradicts to (2) and (iii) contradicts to (1). If (ii) holds, we have $y_1 \leq y_2$ and $p(y_1) = p(y_2)$. Then we have $y_1 = y_2$ from (1) (or (2)) and this contradicts to the assumption of y_1, y_2 . The proof of the fact that the case (iv) contradicts is similar to the case (ii). So we have that $\text{st}(y')$ ($y' \in p^{-1}(y)$) are disjoint.

The relation $p^{-1}(\text{st}(y)) \supset \coprod_{y' \in p^{-1}(y)} \text{st}(y')$ is obvious. Let $\emptyset \neq \sigma \in p^{-1}(\text{st}(y))$ and $z' \in \sigma$. Suppose $y \leq p(z')$. Then there exists $y' \in p^{-1}(y)$ such that $y' \leq z'$. We want to show that $\sigma \in \text{st}(y')$. Let $x' \in \sigma$. If $x' \geq z'$, then $x' \geq y'$ and we have $x' \in V(\text{st}(y'))$. So we assume $x' \leq z'$. Suppose $p(x') \leq y$. Then there exists $x'' \leq y'$ such that $p(x') = p(x'')$. But since $x' \leq z'$ and $x'' \leq z'$, we have $x' = x''$. The case $y \leq p(x')$ is similar. Hence we have $\sigma \in \text{st}(y')$ if $y \leq p(z')$. The case $y \geq p(z')$ is similar, and we have $p^{-1}(\text{st}(y)) \subset \coprod_{y' \in p^{-1}(y)} \text{st}(y')$. \square

Relation to group actions :

Definition 3.11. Let Γ be a group and G a graph and α a right Γ -action on G . α is called a *2-covering action* if $N_2(v) \cap N_2(v\gamma) = \emptyset$ for every $v \in V(G)$ and $\gamma \in \Gamma \setminus \{e_\Gamma\}$. We define similarly that a left Γ -action on a graph is a 2-covering action.

Proposition 3.12. Let G be a graph having no isolated vertices, Γ a group, and α a right Γ -action on G . Consider the following three conditions.

- (1) α is a 2-covering action.
- (2) α is free and the quotient map $p : G \rightarrow G/\Gamma$ is a 2-covering map.
- (3) α is effective and the quotient map $p : G \rightarrow G/\Gamma$ is a 2-covering map.

In any case, (1) and (2) are equivalent. If G is connected, then the above conditions are equivalent.

Proof. (1) \Rightarrow (2) : Since we assume that G has no isolated vertices, we have $v \in N_2(v)$ for every vertex v of G . Therefore $N_2(v) \cap N_2(v\gamma) = \emptyset$ implies $v \neq v\gamma$ and we have the 2-covering action is free.

Let $v \in V(G)$. It is sufficient to show that $p|_{N(v)}$ is surjective and $p|_{N_2(v)}$ is injective (see Lemma 3.3). Let $a \in N(p(v))$. Then there are $w \in a$ and $\gamma_0, \gamma_1 \in \Gamma$ such that $(v\gamma_0, w\gamma_1) \in E(G)$. Therefore we have $w\gamma_1\gamma_0^{-1} \in N(v)$ and $p(w\gamma_1\gamma_0^{-1}) = a$. Hence $p|_{N(v)}$ is surjective. Let $w_0, w_1 \in N_2(v)$ and suppose $p(w_0) = p(w_1)$. Then there exists $\gamma \in \Gamma$ such that $w_0\gamma = w_1$. Since $v \in N_2(w_0) \cap N_2(w_1)$, we have that γ is the identity of Γ from the definition of 2-covering action. Hence $w_0 = w_1$, and we have $p|_{N_2(v)}$ is injective.

(2) \Rightarrow (1) : Let $v \in V(G)$ and $\gamma \in \Gamma$. Suppose $N_2(v) \cap N_2(v\gamma) \neq \emptyset$ and let $w \in N_2(v) \cap N_2(v\gamma)$. Since $v, v\gamma \in N_2(w)$ and $p(v) = p(v\gamma)$ and p is a 2-covering, we have that $v = v\gamma$. Since the Γ -action α of G is free, we have that γ is the identity of Γ . Therefore α is a 2-covering action.

(2) \Rightarrow (3) is obvious. We suppose that G is connected and prove (3) \Rightarrow (2) in this case. Let $v \in V(G)$ and $\gamma \in \Gamma$ and suppose $v = v\gamma$. Then the map $f_\gamma : G \rightarrow G$, $v \mapsto v\gamma$ has a fixed point v . Since G is connected, we have $f_\gamma = \text{id}_G$ from Proposition 3.6. Since the action α is effective, we have γ is the identity of Γ . Hence we have α is free. \square

Example 3.13. (1) Let n and k be positive integers such that $k \geq 3$ and $k \neq 4$. Then, the action of $\mathbb{Z}/n\mathbb{Z}$ on C_{nk} by $x\tau = x + k$, where $\tau = (1 \bmod n) \in \mathbb{Z}/n\mathbb{Z}$, is a 2-covering action, and its quotient graph is C_k . (2) Let k be an integer with $k \geq 3$ and $k \neq 4$. Then the right action of \mathbb{Z} on L defined by $x \cdot n = x + kn$

is a 2-covering action, and its quotient graph is C_k .

(3) For $r \geq 2$, the action of $\mathbb{Z}/2\mathbb{Z}$ on C_{2r} defined by $x\tau = 1 - 2x$, where τ is the generator of $\mathbb{Z}/2\mathbb{Z}$, is a 2-covering action, and its quotient graph has two looped vertices.

(4) For $r \geq 1$, the action of $\mathbb{Z}/2\mathbb{Z}$ on C_{2r+1} defined by $x\tau = 1 - 2x$ is not a 2-covering action, since this action fixes $r \in V(C_{2r+1})$.

Pullbacks of 2-coverings : Let $f : G \rightarrow H$ be a 2-covering map and $p : K \rightarrow H$ be a 2-covering over H . Then we define the graph f^*K by

$$V(f^*K) = \{(x, y) \in V(G) \times V(K) \mid f(x) = p(y)\}$$

$$E(f^*K) = \{((x_1, y_1), (x_2, y_2)) \mid (x_1, x_2) \in E(G) \text{ and } (y_1, y_2) \in E(K)\}.$$

Lemma 3.14. *The projection $q : f^*K \rightarrow G, (x, y) \mapsto x$ is a 2-covering map.*

Proof. Let $(x, y) \in f^*K$. Let $x_0 \in N(x)$. Since $f(x) = p(y)$, there is y' such that $p(y') = x'$. Therefore $(x', y') \in N(x, y)$ and $q(x', y') = x'$. Therefore $q|_{N(x, y)}$ is surjective. Let $(x_0, y_0), (x_1, y_1) \in N_2(x, y)$ with $x_0 = x_1$. Then since $p(y_0) = f(x_0) = f(x_1) = p(y_1)$ and $y_0, y_1 \in N_2(y)$, we have $y_0 = y_1$. Hence $q|_{N_2(x, y)}$ is injective. Therefore q is a 2-covering map. \square

Lemma 3.15. *Let G be a graph having no isolated vertices and n a nonnegative integer. Let i_k denote the graph homomorphism $G \rightarrow G \times I_n, x \mapsto (x, k)$ for $k = 0, 1, \dots, n$. Then for a 2-covering map $p : E \rightarrow G \times I_n$, $i_0^*E \cong i_n^*E$.*

Proof. We can assume that $n = 1$. Let $(x, e) \in i_0^*E$. Since $(x, 1) \in N_2(x, 0)$, there exists $f(e) \in N_2(e)$ such that $p(f(e)) = (x, 0)$. We want to show that f is a graph map. Let $(x', e') \in i_0^*E$ such that $(x, e) \sim (x', e')$. Then $f'(e) \sim e \sim e' \sim f(e)$ and $p(f(e)) = (x, 1) \sim (x', 1) = p(f(e'))$. Hence there is $e'' \in N(f(e))$ such that $p(e'') = p(f(e))$. Since $f(e), e'' \in N_2(e')$, we have $f(e) = e''$ and $f(e) \sim f(e')$. \square

Proposition 3.16. *Let G, H be graphs having no isolated points and $f, g : G \rightarrow H$ be graph homomorphisms with $f \simeq g$ and $p : E \rightarrow H$ be a 2-covering map. Then $f^*E \cong g^*E$ as a 2-covering over G .*

Proof. $F : G \times I_n \rightarrow H$ be a \times -homotopy from f to g . Then $f^*E \cong i_0^*F^*E \cong i_n^*F^*E \cong g^*E$. \square

4. 2-FUNDAMENTAL GROUPS

In this section, we give the definition of a 2-fundamental group of a based graph and study its basic property. After that, we investigate the relation between based 2-covering maps and 2-fundamental groups.

Definition of 2-fundamental groups :

Let n be a nonnegative integer. The graph L_n is defined by $V(L_n) = \{0, 1, \dots, n\}$ and $E(L_n) = \{(x, y) \mid |x - y| = 1\}$. A graph homomorphism from L_n to a graph G is called a *path* of G with length n . Given a path φ of G , the length of φ is denoted by $l(\varphi)$ and $\varphi(0)$ is called the *initial point* of φ and $\varphi(l(\varphi))$ is called the *terminal point* of φ . For vertices $v, w \in V(G)$, a path from v to w is a path whose initial point is v and whose terminal point is w . We denote the set of all paths from v to w by $P(G; v, w)$.

We consider the following two conditions for $(\varphi, \psi) \in P(G; v, w) \times P(G; v, w)$.

- (i) $l(\varphi) + 2 = l(\psi)$ and there exists $x \in \{0, 1, \dots, l(\varphi)\}$ such that $\varphi(i) = \psi(i)$ for $i \leq x$ and $\varphi(i+2) = \psi(i)$ for $i \geq x$.
- (ii) $l(\varphi) = l(\psi)$ and $\varphi \times \psi(E(L_{l(\varphi)})) \subset E(G)$.

We write \simeq for the equivalence relation on $P(G; v, w)$ generated by the above two conditions.

Remark 4.1. Consider the next condition for $(\varphi, \psi) \in P(G; v, w) \times P(G; v, w)$.

- (ii)' $l(\varphi) = l(\psi)$ and there exists $x \in \{0, 1, \dots, l(\varphi)\}$ such that $\varphi(i) = \psi(i)$ for all $i \neq x$.

Then the equivalence relation \simeq' generated by (i) and (ii)' is equal to \simeq . In fact, if (φ, ψ) satisfies the condition (ii)', then (φ, ψ) satisfies the condition (ii). Thus $\varphi \simeq' \psi$ implies $\varphi \simeq \psi$. If (φ, ψ) satisfies the condition (ii), let η_k for $1 \leq k \leq l(\varphi)$ denote by the graph homomorphism defined by $\eta_k(i) = \psi(i)$ for $i < k$ and $\eta_k(i) = \varphi(i)$ for $i \geq k$. Since (η_k, η_{k-1}) satisfies the condition (ii)' and $\eta_0 = \varphi$ and $\eta_{l(\varphi)} = \psi$, we have $\varphi \simeq' \psi$. Hence we have that \simeq' is equal to \simeq .

We write $\pi_1^2(G; v, w)$ for the quotient set $P(G; v, w) / \simeq$. For $\varphi, \psi \in P(G; v, w)$, we say that φ is 2-homotopic to ψ if $\varphi \simeq \psi$. For $\varphi \in P(G; v, w)$, the equivalence class of \simeq represented by φ with respect to \simeq is denoted by $[\varphi]$, and is called the 2-homotopy class of φ .

For $\varphi : L_n \rightarrow G$ and $\psi : L_m \rightarrow G$ such that $\varphi(n) = \psi(0)$, we define the composition of φ and ψ by the path $\psi \cdot \varphi : L_{n+m} \rightarrow G$ such that $(\psi \cdot \varphi)(i) = \varphi(i)$ ($i \leq n$) and $(\psi \cdot \varphi)(i) = \psi(i - n)$ ($i \geq n$). Obviously this operation is associative, i.e. $(\varphi \cdot \psi) \cdot \eta = \varphi \cdot (\psi \cdot \eta)$ if these are composable.

Lemma 4.2. *Let G be a graph and $u, v, w \in V(G)$ and $\varphi, \varphi' \in P(G; v, w), \psi, \psi' \in P(G; u, v)$. If $\varphi \simeq \varphi'$ and if $\psi \simeq \psi'$, then $\varphi \cdot \psi \simeq \varphi' \cdot \psi'$.*

Proof. We can assume $\varphi = \varphi'$ or $\psi = \psi'$. Suppose $\varphi = \varphi'$. We can assume (ψ, ψ') satisfies the condition (i) in the definition of \simeq or (ii)' in Remark 4.1. But in this case $\varphi \cdot \psi \simeq \varphi \cdot \psi'$ is obvious. The case $\psi = \psi'$ is similar. \square

From Lemma 4.2, the composition of paths induces the composition of 2-homotopy classes of paths

$$\pi_1^2(G; v, w) \times \pi_1^2(G; u, w) \longrightarrow \pi_1^2(G; u, w), \quad ([\varphi], [\psi]) \mapsto [\varphi \cdot \psi].$$

We write $\alpha \cdot \beta$ for the composition of 2-homotopy classes, for $\alpha \in \pi_1^2(G; v, w)$ and for $\beta \in \pi_1^2(G; u, v)$.

Given a graph G and $v \in V(G)$, then the path $L_0 \rightarrow G$, $0 \mapsto v$ is denoted by $*_v$. It is obvious that $*_v \cdot \varphi = \varphi$ and $\psi \cdot *_v = \psi$ if composable.

Let $\varphi : L_n \rightarrow G$ be a path of a graph G . The path $L_n \rightarrow G$ defined by $i \mapsto \varphi(n - i)$ is denoted by $\overline{\varphi}$. From the condition (i), we have $\overline{\varphi} \cdot \varphi \simeq *_v$ and $\varphi \cdot \overline{\varphi} \simeq *_w$, where v is the initial point of φ and w is the terminal point of φ .

Let G be a graph and φ, ψ paths of G . If $\varphi \simeq \psi$, then we have $l(\varphi) = l(\psi) \pmod{2}$ from the definition of 2-homotopy. Thus we say that the 2-homotopy class α is *even* if the length of the representative of α is even, and we say that α is *odd* if α is not even.

A *loop* of a based graph (G, v) is a path of G from v to v .

Definition 4.3. For a based graph (G, v) , $\pi_1^2(G; v, v)$ is denoted by $\pi_1^2(G, v)$. $\pi_1^2(G, v)$ is a group with the composition and is called the *2-fundamental group* of (G, v) .

Obviously, the identity element of $\pi_1^2(G, v)$ is $[\ast_v]$, and the inverse element of $[\varphi] \in \pi_1(G, v)$ is $[\bar{\varphi}]$.

In this paper, we only consider the 2-fundamental groups. Hence we write $\pi_1(G, v)$ for $\pi_1^2(G, v)$ and $\pi_1(G; v, w)$ for $\pi_1^2(G; v, w)$.

All even elements of $\pi_1(G, v)$ forms a subgroup of $\pi_1(G, v)$ and we write $\pi_1(G, v)_{\text{ev}}$ for this subgroup. $\pi_1(G, v)_{\text{ev}}$ is called the *even part of the 2-fundamental group* of (G, v) . $\pi_1(G, v)_{\text{ev}}$ is a normal subgroup of $\pi_1(G, v)$ whose index is 1 or 2. Indeed $\pi_1(G, v)_{\text{ev}}$ is the kernel of the group homomorphism

$$\pi_1(G, v) \longrightarrow \mathbb{Z}/2\mathbb{Z}, \quad [\varphi] \longmapsto (l(\varphi) \bmod 2).$$

We remark that $\pi_1(G, v)_{\text{ev}} = \pi_1(G, v)$ if and only if the connected component of G containing v is 2-colorable. In Section 6, we will prove that $\pi_1(G, v)_{\text{ev}}$ is isomorphic to the fundamental group of $(|\mathcal{N}(G)|, v)$ if v is not isolated.

Let $f : G \rightarrow H$ be a graph homomorphism and let $\varphi, \psi \in P(G; v, w)$ where v, w are vertices of G . We can easily see that $f \circ \varphi \simeq f \circ \psi$ if $\varphi \simeq \psi$. If φ and ψ are composable, then $f \circ (\varphi \cdot \psi) = (f \circ \varphi) \cdot (f \circ \psi)$. Therefore a based graph map $f : (G, v) \rightarrow (H, w)$ induces a group homomorphism $\pi_1(G, v) \rightarrow \pi_1(H, w)$, and is denoted by $\pi_1(f)$ or f_* . Obviously $\pi_1(\text{id}) = \text{id}$ and $\pi_1(g \circ f) = \pi_1(g) \circ \pi_1(f)$ and these preserve even parts. Therefore π_1 and the even part of π_1 are functors from the category of based graphs (**Graphs**)_{*} to the category of groups (**Groups**).

We investigate how $\pi_1(f)$ depends on the choice of a based graph homomorphism f . Then we need to consider the \times -homotopy for the based graph case. Dochtermann defined the based case of \times -homotopy in [4], but he assumed that basepoint has a loop.

For based graph homomorphisms $f, g : (G, v) \rightarrow (H, w)$, we say that f is \times -homotopic to g if there exists a nonnegative integer n and a graph homomorphism $F : G \times I_n \rightarrow H$ such that $F(v, i) = w$ for all $0 \leq i \leq n$ and $F(x, 0) = f(x)$ and $F(x, n) = g(x)$ for all $x \in V(G)$.

Let (G, v) and (H, w) be based graphs. A *based multihomomorphism* from (G, v) to (H, w) is $\eta \in \text{Hom}(G, H)$ such that $\eta(v) = \{w\}$. The poset of all based multihomomorphisms from (G, v) to (H, w) is denoted by $\text{Hom}((G, v), (H, w))$. Then we can easily see that, for based graph homomorphisms $f, g : (G, v) \rightarrow (H, w)$, f is \times -homotopic to g in the based sense if and only if f and g are contained in the same connected component of $\text{Hom}((G, v), (H, w))$.

We remark that $\text{Hom}((G, v), (H, w))$ does not have a canonical basepoint. Indeed $\text{Hom}((G, v), (H, w))$ may be empty.

Lemma 4.4. *Let $f, g : (G, v) \rightarrow (H, w)$ be based graph homomorphisms. If f is \times -homotopic to g in the based sense, then we have $\pi_1(f) = \pi_1(g)$.*

Proof. We can assume that there is $\eta \in \text{Hom}((G, v), (H, w))$ such that $f \leq \eta$ and $g \leq \eta$. In this case $f \times g(E(G)) \subset E(H)$. Therefore for a loop $\varphi : L_n \rightarrow G$,

$$(f \circ \varphi) \times (g \circ \varphi)(E(L_n)) = (f \times g) \circ (\varphi \times \varphi)(E(L_n)) \subset E(H)$$

and we have $f \circ \varphi \simeq g \circ \varphi$. □

Let G be a graph and $v, w \in V(G)$, $\alpha \in \pi_1(G; v, w)$. We define the group homomorphism $\text{Ad}(\alpha) : \pi_1(G, v) \rightarrow \pi_1(G, w)$ by $\text{Ad}(\alpha)(\beta) = \alpha \cdot \beta \cdot \bar{\alpha}$. We write $\text{Ad}(\varphi) = \text{Ad}([\varphi])$ for $\varphi \in P(G; v, w)$. We remark that $\text{Ad}(\alpha)$ preserves even parts.

From the definition of Ad , we can easily see that $\text{Ad}([*_v]) = \text{id}_{\pi_1(G,v)}$ and $\text{Ad}(\beta) \circ \text{Ad}(\alpha) = \text{Ad}(\beta \cdot \alpha)$ for $\beta \in \pi_1(G; v, w)$ and $\alpha \in \pi_1(G; u, v)$. Therefore $\text{Ad}(\bar{\alpha}) \circ \text{Ad}(\alpha) = \text{id}$ and $\text{Ad}(\alpha) \circ \text{Ad}(\bar{\alpha}) = \text{id}$. So we have $\text{Ad}(\alpha)$ is an isomorphism and $\text{Ad}(\alpha)^{-1} = \text{Ad}(\bar{\alpha})$. Thus if G is connected, the group $\pi_1(G, v)$ is, up to isomorphisms, independent of the choice of the basepoint v . In this case, the notation $\pi_1(G, v)$ is often abbreviated to $\pi_1(G)$.

Let (G, v) be a based graph and $\alpha \in \pi_1(G, v)$. In this case, $\text{Ad}(\alpha)$ is an inner automorphism of $\pi_1(G, v)$ with respect to α .

Let $\alpha, \beta \in \pi_1(G; v, w)$. In general, $\text{Ad}(\alpha)$ is not equal to $\text{Ad}(\beta)$. However, we have that $\beta \cdot \bar{\alpha} \in \pi_1(G, w)$ and $\text{Ad}(\beta) = \text{Ad}(\beta \cdot \bar{\alpha}) \circ \text{Ad}(\alpha)$. Since an inner automorphism fixes the abelianization, the abelian group homomorphisms $\pi_1(G, v)/[\pi_1(G, v), \pi_1(G, v)] \rightarrow \pi_1(G, w)/[\pi_1(G, w), \pi_1(G, w)]$ induced by $\text{Ad}(\alpha)$ is independent of the choice of α .

We can define the functor H_1 from the category of graphs (**Graphs**) to the category of abelian groups (**Abels**) as follows. Let G be a graph. Let $G = \coprod G_\alpha$ be a decomposition of connected components of G . We choose one vertex v_α for each connected component G_α of G , and we define $H_1(G) = \bigoplus \pi_1(G, v_\alpha)/[\pi_1(G, v_\alpha), \pi_1(G, v_\alpha)]$. Then the functor H_1 is independent of the choice of v_α up to natural isomorphisms.

Proposition 4.5. *Let G, H be graphs and f, g graph homomorphisms from G to H . Let $v \in V(G)$ be a non-isolated vertex. If $f \simeq_\times g$ then there exists a path γ from $f(v)$ to $g(v)$ such that*

$$\pi_1(g) = \text{Ad}(\gamma) \circ \pi_1(f) : \pi_1(G, v) \rightarrow \pi_1(H, g(v))$$

Therefore if G has no isolated vertices and if $f \simeq_\times g$, then $H_1(f) = H_1(g)$.

Proof. Let $F : G \times I_n \rightarrow G$ be a \times -homotopy from f to g . Since v is not isolated, there is $w \in N(v)$. Define a path $\gamma' : L_{2n} \rightarrow G \times I_n$ by $\gamma'(2i) = (v, i)$ ($0 \leq i \leq n$) and $\gamma'(2i-1) = (w, i)$ ($1 \leq i \leq n$). Set $\gamma = F \circ \gamma'$. We prove $\pi_1(g) = \text{Ad}(\gamma) \circ \pi_1(f)$.

Let k be an integer such that $0 \leq k \leq n$. Let γ'_k denote $\gamma'|_{L_{2k}}$ and i_k denote the inclusion $G \rightarrow G \times I_n$, $x \mapsto (x, k)$. Let $\varphi : L_n \rightarrow G$ be a loop of (G, v) and set $\varphi_k = \overline{\gamma'_k} \cdot \varphi'_k \cdot \gamma'_k$. Firstly, we prove $\varphi_n \simeq \varphi_0$. It is sufficient to show that $\varphi_k \simeq \varphi_{k-1}$ for $1 \leq k \leq n$. Define a path $\psi_k : L_{4k+n} \rightarrow G \times I_n$ by $\psi_k(i) = \varphi_k(i)$ ($i \leq 2k-2$ or $i \geq 2k+n+2$) and $\psi_k(2k-1) = \psi_k(2k+n+1) = (w, k-1)$, $\psi_k(2k) = (v, k-1) = \psi_k(2k+n)$, $\psi_k(x) = (\varphi(x), k-1)$ ($2k \leq x \leq 2k+n$). We can easily see $(\varphi_k \times \psi_k)(E(L_n)) \subset E(G \times I_n)$. From (i) in the definition of 2-homotopy of paths, $\psi_k \simeq \varphi_{k-1}$ and we prove $\varphi_n \simeq \varphi_0$.

Therefore we have

$$\text{Ad}(\bar{\gamma}) \circ g_*([\varphi]) = F_*([\overline{\gamma'} \cdot (i_n \circ \varphi) \cdot \gamma']) = [F \circ \varphi_n] = [F \circ \varphi_0] = f_*[\varphi].$$

Hence $g_* = \text{Ad}(\gamma) \circ f_*$. □

Corollary 4.6. *Let G, H be graphs and f a graph homomorphism from G to H , $v \in V(G)$ a non-isolated vertex. If f is \times -homotopy equivalence, $f_* : \pi_1(G, v) \rightarrow \pi_1(H, f(v))$ is an isomorphism.*

Proof. Let g be a \times -homotopy inverse of f . Since $g \circ f \simeq_\times \text{id}$, we have $g_* \circ f_*$ is an isomorphism from previous proposition. Hence we have f_* is injective. On the other hand, since $f \circ g \simeq \text{id}$, we have $f_* \circ g_*$ is an isomorphism. Hence f_* is surjective. □

Relation to 2-covering maps :

A based graph homomorphism $p : (G, v) \rightarrow (H, w)$ is called a *based 2-covering map* or a *based 2-covering over* (H, w) if $p : G \rightarrow H$ is 2-covering map. A based 2-covering $p : (G, v) \rightarrow (H, w)$ over (H, w) is called *connected* if G is connected.

Lemma 4.7. *Let $p : (G, v) \rightarrow (H, w)$ be a based covering map.*

- (1) *Let $\varphi : (L_n, 0) \rightarrow (H, w)$. Then there exists a unique $\tilde{\varphi} : (L_n, 0) \rightarrow (G, v)$ with $\varphi = p\tilde{\varphi}$.*
- (2) *We write $\tilde{\varphi}$ for a graph homomorphism $(L_n, 0) \rightarrow (G, v)$ such that $\varphi = p\tilde{\varphi}$ for a based graph homomorphism $\varphi : (L_n, 0) \rightarrow (H, w)$. Let $u \in V(G)$ and $\varphi, \psi \in P(G; w, u)$. If $\varphi \simeq \psi$, then the terminal point of $\tilde{\varphi}$ is equal to the terminal point of $\tilde{\psi}$, and we have $\tilde{\varphi} \simeq \tilde{\psi}$.*

Proof. Since the case of $n = 0$ is obvious, we assume $n \geq 1$. Therefore L_n has no isolated points.

(1) The uniqueness of $\tilde{\varphi}$ follows from Proposition 3.6. Let $x_0 = v$. Let $x_1 \in N(x_0)$ such that $\varphi(1) \in N(\varphi(0)) = N(w)$. By induction, we obtain a sequence (x_0, \dots, x_n) of vertices of G such that $x_0 = v$ and $p(x_i) = \varphi(i)$. Set $\tilde{\varphi}(i) = x_i$ for $0 \leq i \leq n$, then $\tilde{\varphi}$ is the lift of φ .

(2) We can assume that φ and ψ satisfy the condition (i) or (ii) in the definition of 2-homotopy of paths.

Suppose φ and ψ satisfy the condition (i) and let n be the length of φ . Then $l(\psi) = n + 2$ and there is $x \in \{0, \dots, n\}$ such that $\varphi(i) = \psi(i)$ for $i \leq x$ and $\varphi(i) = \psi(i + 2)$ for $i \geq x$. From the uniqueness of (1), we have $\tilde{\varphi}(i) = \tilde{\psi}(i)$ for $i \leq x$. Since $\tilde{\psi}(x)$ and $\tilde{\psi}(x + 2)$ are elements of $N(\tilde{\psi}(x))$ and $p(\tilde{\psi}(x)) = \psi(x) = \psi(x + 2) = p(\tilde{\psi}(x + 2))$, we have $\tilde{\psi}(x + 2) = \tilde{\psi}(x) = \tilde{\varphi}(x)$. From the uniqueness of (1), we have $\tilde{\varphi}(i) = \tilde{\psi}(i + 2)$ for $x \leq i$. Therefore $\tilde{\varphi}$ and $\tilde{\psi}$ satisfy the condition (i).

Suppose φ and ψ satisfy the condition (ii). Then the map $F : V(L_n \times I_1) \rightarrow V(H)$ where $F(x, 0) = \varphi(x)$ and $F(x, 1) = \psi(x)$ is a graph map. From Corollary 3.8, we have a graph map $\tilde{F} : L_n \times I_1 \rightarrow G$ such that $\tilde{F}(x, 0) = \tilde{\varphi}(x)$ and $p\tilde{F} = F$. Since $\tilde{F}(x, 1) \in N_2(\tilde{F}(x, 0))$ and $p\tilde{F}(0, 0) = p\tilde{F}(0, 1)$ and $p\tilde{F}(n, 0) = p\tilde{F}(n, 1)$, we have $\tilde{F}(0, 0) = \tilde{F}(0, 1)$ and $\tilde{F}(n, 0) = \tilde{F}(n, 1)$. Hence we have $\tilde{\psi}(x) = \tilde{F}(x, 1)$ and $\tilde{\psi}(n) = \tilde{\varphi}(n)$. \square

Corollary 4.8. *Let $p : (G, v) \rightarrow (H, w)$ be a based 2-covering map. Then $\pi_1(p)$ is injective. Let φ be a loop of (H, w) . Then $[\varphi] \in p_*(\pi_1(G, v))$ if and only if the lift $\tilde{\varphi}$ of φ with respect to $p : (G, v) \rightarrow (H, w)$ whose initial point is v is a loop of (G, v) .*

Proof. Let ψ be a loop of (G, v) with $p\psi \simeq *_{w}$. Since $*_v$ is the lift of $*_w$, we have $\psi \simeq *_v$ from the previous lemma. Therefore $\pi_1(f) : \pi_1(G, v) \rightarrow \pi_1(H, w)$ is injective.

Let φ be a loop of (H, w) . If $\tilde{\varphi}$ is a loop of (G, v) , then we obviously have $[\varphi] \in p_*\pi_1(G, v)$. Suppose $[\varphi] \in p_*\pi_1(G, v)$. Then there exists a loop ψ of (G, v) such that $p\psi \simeq \varphi$. From the previous lemma, the terminal point of $\tilde{\varphi}$ is equal to the terminal point of ψ . So $\tilde{\varphi}$ is a loop of (G, v) . \square

Proposition 4.9. *Let $p : (G, v) \rightarrow (H, w)$ be a based 2-covering and (T, x) a connected based graph and $f : (T, x) \rightarrow (H, w)$ a based graph map. Then there exists a graph map $\tilde{f} : (T, x) \rightarrow (G, v)$ such that $p\tilde{f} = f$ if and only if $f_*(\pi_1(T, x)) \subset p_*(\pi_1(G, v))$.*

Proof. Suppose there exists $\tilde{f} : (T, x) \rightarrow (G, v)$ such that $p\tilde{f} = f$. Then we have $f_*(\pi_1(T, x)) = p_*\tilde{f}_*(\pi_1(T, x)) \subset p_*\pi_1(G, v)$.

Suppose $f_*\pi_1(T, x) \subset p_*\pi_1(G, v)$. We define the graph map $\tilde{f} : (T, x) \rightarrow (G, v)$ as follows. Let $y \in V(T)$, and φ a path of T from x to y . Let $\tilde{\varphi}$ be the lift of $f\varphi$ whose initial point is v . We want to

define $\tilde{f}(y)$ by the terminal point of $\tilde{\varphi}$. Let ψ be another path of T from x to y , and $\tilde{\psi}$ the lift of $f\psi$ with respect to p whose initial point is v . We want to show the terminal point of $\tilde{\psi}$ is equal to $\tilde{\psi}$. Since $f_*\pi_1(T, x) \subset p_*\pi_1(G, v)$, the lift of $f(\overline{\psi} \cdot \varphi)$ is a loop of (G, v) . We write γ for the lift of $f(\overline{\psi} \cdot \varphi)$. Since $f(\psi \cdot \overline{\psi} \cdot \varphi) \simeq f(\varphi)$, the terminal point of the lift of $f(\psi \cdot \overline{\psi} \cdot \varphi)$ is equal to the terminal point of $\tilde{\varphi}$. Since the lift of $f(\psi \cdot \overline{\psi} \cdot \varphi)$ is $\tilde{\psi} \cdot \gamma$, we have that the terminal point of $\tilde{\psi}$ is equal to the terminal point of $\tilde{\varphi}$ from Lemma 4.7. Therefore \tilde{f} is well-defined as a set map.

We want to show that \tilde{f} is a graph homomorphism. Let $(y, z) \in E(T)$, and $\varphi : L_n \rightarrow T$ be a path from x to y . We define the path $\varphi' : L_{n+1} \rightarrow T$ by $\varphi'|_{L_n} = \varphi$ and $\varphi'(n+1) = z$. Let $\tilde{\varphi}$ be the lift of $f\varphi$ whose initial point is v . Since $f(z) \in N(f(y)) = N(p(\varphi(n)))$, there exists $z' \in N(\tilde{\varphi}(n))$ with $p(z') = f(z) = f(\varphi'(n+1))$. Then the lift $\tilde{\varphi}' : L_{n+1} \rightarrow G$ whose initial point is v is obtained by $\tilde{\varphi}'|_{L_n} = \tilde{\varphi}$ and $\varphi'(n+1) = z'$. Therefore $\tilde{f}(z) = z' \in N(\tilde{f}(y))$ and we have \tilde{f} is a graph homomorphism. \square

Remark 4.10. From Proposition 3.6 the lift \tilde{f} in the previous proposition is unique.

Lemma 4.11. *Let $p : (G, v) \rightarrow (H, w)$ be a connected based 2-covering map. Then there exists a bijection*

$$\Phi : \pi_1(H, w)/p_*\pi_1(G, v) \cong p^{-1}(w).$$

This bijection Φ is obtained as follows. Let φ be a loop of (H, w) and $\tilde{\varphi}$ a lift of φ . Then we set $\Phi([\varphi])$ be the terminal point of $\tilde{\varphi}$.

Proof. Firstly, we want to show that the map Φ is well-defined. Let φ, ψ be loops of (H, w) with $[\varphi] = [\psi]$ in $\pi_1(H, w)/p_*\pi_1(G, v)$. Then there is a loop γ of (H, w) such that the lift $\tilde{\gamma}$ is a loop of (G, v) and $\varphi \simeq \psi \cdot \gamma$. From Lemma 4.7, the terminal point of $\tilde{\varphi}$ is equal to the terminal point of the lift of $\psi \cdot \gamma$. But the lift of $\psi \cdot \gamma$ is $\tilde{\psi} \cdot \tilde{\gamma}$ since $\tilde{\gamma}$ is a loop. Therefore Φ is well-defined.

Since (G, v) is connected, Φ is surjective. Let φ, ψ be loops of (H, w) with $\Phi([\varphi]) = \Phi([\psi])$. Then $\tilde{\varphi} \cdot \tilde{\psi}$ is a loop of (G, v) and $\psi \simeq \varphi \cdot \overline{\varphi} \cdot \psi \simeq \varphi \cdot p(\tilde{\varphi} \cdot \tilde{\psi})$. Therefore $[\varphi] = [\psi]$ in $\pi_1(H, w)/p_*\pi_1(G, v)$. Hence Φ is injective. \square

Let (G, v) be a based graph. A connected based 2-covering map $(\tilde{G}, \tilde{v}) \rightarrow (G, v)$ is called a *universal 2-covering over (G, v)* if $\pi_1(\tilde{G}, \tilde{v})$ is trivial. We deduce that the universal 2-covering over (G, v) is unique up to isomorphism over (G, v) from Proposition 4.9 and Remark 4.10.

The following corollary is useful to compute 2-fundamental groups.

Corollary 4.12. *Let (G, v) be a connected based graph and $p : (\tilde{G}, \tilde{v}) \rightarrow (G, v)$ be a universal 2-covering over (G, v) . For each $x \in p^{-1}(v)$, let φ_x be a path from \tilde{v} to x . Then $[p \circ \varphi_x] \neq [p \circ \varphi_y]$ for $x \neq y$, and $\pi_1(G, v) = \{[p \circ \varphi_x] \mid x \in p^{-1}(v)\}$.*

Proposition 4.13. *Let (G, v) be a graph. Then there exists a universal 2-covering over (G, v) .*

Proof. Let $V(\tilde{G}) = \coprod_{w \in V(G)} \pi_1(G; v, w)$, $E(\tilde{G}) = \{(\alpha, \beta) \mid \text{There is } \varphi \in \beta \text{ such that } \varphi|_{L_{l(\varphi)-1}} \in \alpha\}$, $p(\pi_1(G; v, w)) \subset \{w\}$ and $\tilde{v} = [*_v]$.

We show that $\tilde{G} = (V(\tilde{G}), E(\tilde{G}))$ is a graph. Firstly, we claim that the following three conditions for $(\alpha, \beta) \in V(\tilde{G}) \times V(\tilde{G})$ is equivalent.

- (1) $(\alpha, \beta) \in E(\tilde{G})$.
- (2) For each $\varphi \in \alpha$, the map $\varphi' : V(L_{l(\varphi)+1}) \rightarrow V(G)$ defined by $\varphi'|_{V(L_{l(\varphi)})} = \varphi$ and $\varphi'(l(\varphi) + 1) = p(\beta)$

is a graph homomorphism and an element of β .

(3) There exists $\varphi' \in \alpha$ such that the map $\varphi' : V(L_{l(\varphi)+1}) \rightarrow V(G)$ defined by $\varphi'|_{V(L_{l(\varphi)})} = \varphi$ and $\varphi'(l(\varphi) + 1) = p(\beta)$ is a graph homomorphism and an element of β .

In fact (2) \Rightarrow (3) \Leftrightarrow (1) is obvious, and (3) \Rightarrow (2) is deduced from Lemma 4.2.

Let $(\alpha, \beta) \in E(\tilde{G})$, the path $\varphi \in \alpha$ and $n = l(\varphi)$. Then φ' defined above is an element of β . Let $\varphi'' : L_{n+2} \rightarrow G$ denote the path defined by $\varphi''|_{L_{n+1}} = \varphi'$ and $\varphi''(n+2) = \varphi(n) = p(\alpha)$. We have $\varphi'' \in \alpha$ since $\varphi'' \simeq \varphi$. Therefore $(\beta, \alpha) \in E(\tilde{G})$ and $\tilde{G} = (V(\tilde{G}), E(\tilde{G}))$ is a graph.

We show p is a 2-covering map of graphs. It is obvious that p is a graph homomorphism. Let $\alpha \in V(\tilde{G})$ and $x = p(\alpha)$. Let $y \in N(x)$ and $\varphi \in \alpha$ a path with length n . Let $\varphi' : L_{n+1} \rightarrow G$ be a graph map defined by $\varphi'|_{L_n} = \varphi$ and $\varphi'(n+1) = y$. Then we have $[\varphi'] \in N(\alpha)$ from the definition of $E(\tilde{G})$ and $p([\varphi']) = y$. Therefore $p|_{N(\alpha)} : N(\alpha) \rightarrow N(x)$ is surjective. Let $\beta_1, \beta_2 \in N_2(\alpha)$ such that $p(\beta_1) = p(\beta_2) = z$. Then there are $\gamma_1, \gamma_2 \in N(\alpha)$ such that $\beta_i \in N(\gamma_i)$ for $i = 1, 2$. Let $\varphi \in \alpha$ and n be the length of φ . Let $\varphi'_i : L_{n+1} \rightarrow G$ for $i = 1, 2$ be a path defined by $\varphi'_i|_{L_n} = \varphi$ and $\varphi'_i(n+1) = p(\gamma_i)$. Let $\varphi''_i : L_{n+2} \rightarrow G$ for $i = 1, 2$ be a path defined by $\varphi''_i|_{L_{n+1}} = \varphi'_i$ and $\varphi''_i(n+2) = z$. Then $\varphi'_i \in \gamma_i$ and $\varphi''_i \in \beta_i$. Since φ''_1 and φ''_2 satisfy the condition (ii) in the definition of homotopy, we have $\beta_1 = \beta_2$ hence $p|_{N_2(\alpha)}$ is injective. From Lemma 3.3, we obtain that p is a 2-covering map.

We show that \tilde{G} is connected. We remark that, for a based graph map $(L_n, 0) \rightarrow (G, v)$, the map $\tilde{\varphi} : V(L_n) \rightarrow V(\tilde{G})$ defined by $\tilde{\varphi}(i) = [\varphi|_{L_i}]$ is a lift of φ , and $\tilde{\varphi}$ is a path from \tilde{v} to $[\varphi]$. Therefore \tilde{G} is connected.

Finally, we show that $\pi_1(\tilde{G}, \tilde{v})$ is trivial. Since p is a 2-covering map, it is sufficient to show that $p_*\pi_1(\tilde{G}, \tilde{v})$ is trivial. Let $[\varphi] \in p_*\pi_1(\tilde{G}, \tilde{v})$. Then the terminal point of the lift $\tilde{\varphi}$ of φ with respect to p is $\tilde{v} = [*_v]$ from Corollary 4.8. But since the terminal point of $\tilde{\varphi}$ is $[\varphi]$ from the construction of $\tilde{\varphi}$, we have $[\varphi] = [*_v]$. Therefore $p_*\pi_1(\tilde{G}, \tilde{v})$ is trivial. \square

Lemma 4.14. *Let (G, v) be a graph and (\tilde{G}, \tilde{v}) the universal 2-covering over (G, v) constructed in the proof of Proposition 4.13. Then the action*

$$V(\tilde{G}) \times \pi_1(G, v) \rightarrow V(\tilde{G}), (\alpha, \beta) \mapsto \alpha \cdot \beta$$

is a 2-covering action.

Proof. Let $\alpha \in \pi_1(G, v)$ and $(\beta_1, \beta_2) \in E(\tilde{G})$. Then there is $\varphi \in \beta_2$ such that $\varphi|_{L_{n-1}} \in \beta_1$ where n is the length of φ . Let $\psi \in \alpha$ and m be the length of ψ . Then $\varphi \cdot \psi \in \beta_2 \cdot \alpha$ and $\varphi \cdot \psi|_{L_{m+n-1}} = \psi|_{L_{n-1}} \cdot \varphi \in \beta_1 \cdot \alpha$. Therefore $(\beta_1 \cdot \alpha, \beta_2 \cdot \alpha) \in E(\tilde{G})$ from the definition of $E(\tilde{G})$.

Since $\tilde{v} \cdot \alpha = \alpha$ for each $\alpha \in \pi_1(G, v)$, this action is effective and the quotient $\tilde{G} \rightarrow \tilde{G}/\pi_1(G, v) \cong G$ is a 2-covering, this action is a 2-covering action from Proposition 3.12. \square

Proposition 4.15. *Let (G, v) be a based graph and Γ a subgroup of $\pi_1(G, v)$. Then there exists a connected based 2-covering $p_\Gamma : (G_\Gamma, v_\Gamma) \rightarrow (G, v)$ such that $p_{\Gamma*}\pi_1(G_\Gamma, v_\Gamma) = \Gamma$.*

Proof. Let $p : (\tilde{G}, \tilde{v}) \rightarrow (G, v)$ be the universal 2-covering over (G, v) constructed in the proof of Proposition 4.12. We define G_Γ by \tilde{G}/Γ and v_Γ is the image of \tilde{v} of the quotient map $q : \tilde{G} \rightarrow G_\Gamma$. From the universality of a quotient map, we have a graph homomorphism $p_\Gamma : G_\Gamma \rightarrow G$ such that $p_\Gamma \circ q = p$. Since q is a surjective 2-covering map, p_Γ is a 2-covering map from Lemma 3.4.

We want to show that $p_{\Gamma*}\pi_1(G_\Gamma, v_\Gamma)$ is Γ . Let $[\varphi] \in \Gamma$ and n be the length of φ . Since the lift $\tilde{\varphi} : (L_n, 0) \rightarrow (\tilde{G}, \tilde{v})$ of φ is defined by $\tilde{\varphi}(i) = [\varphi|_{L_i}]$, we have $\tilde{\varphi}(n) = [\varphi] \in \Gamma$. Therefore $q\tilde{\varphi}$ is a loop of (G_Γ, v_Γ) . So we have $[\varphi] \in p_{\Gamma*}\pi_1(G_\Gamma, v_\Gamma)$.

Suppose $[\varphi] \in p_*\pi_1(G_\Gamma, v_\Gamma)$. Then $q\tilde{\varphi}$ is the lift of φ of (G_Γ, v_Γ) where $\tilde{\varphi}$ is the lift of φ with respect to $(\tilde{G}, \tilde{v}) \rightarrow (G, v)$. We have the terminal point of $\tilde{\varphi}$ is an element of Γ . So we have $[\varphi] \in \Gamma$. \square

The next theorem summarizes this section.

Theorem 4.16. *Let (G, v) be a based graph. Let \mathcal{C} denote the category whose objects are connected based 2-coverings over (G, v) and morphisms are based graph homomorphisms over (G, v) . Let \mathcal{D} denote the small category whose objects are subgroups of $\pi_1(G, v)$ and the morphisms are inclusions. Then the functor $F : \mathcal{C} \rightarrow \mathcal{D}$, $(p : (\tilde{G}, \tilde{v}) \rightarrow (G, v)) \mapsto \text{Im}(p_* : \pi_1(\tilde{G}, \tilde{v}) \rightarrow \pi_1(G, v))$ is a categorical equivalence.*

Before giving the proof, we recall some terminologies of the category theory. Let F be a functor from a category \mathcal{C} to a category \mathcal{D} . F is said to be *essentially surjective* if, for each object X of \mathcal{D} , there exists an object A of \mathcal{C} such that $FA \cong X$. F is said to be *fully faithful* if, for objects A, B of \mathcal{C} , the map $\mathcal{C}(A, B) \rightarrow \mathcal{D}(FA, FB)$ is bijective. It is known that F is an equivalence of categories if and only if F is fully faithful and essentially surjective. (see [11])

Proof. It is obvious that F is essentially surjective from Proposition 4.15. Let $p_i : (G_i, v_i) \rightarrow (G, v)$ for $i = 1, 2$ be connected based covering. From the Proposition 4.9, there exists a morphism of \mathcal{C} from p_1 to p_2 if and only if $p_{1*}\pi_1(G_1, v_1) \subset p_{2*}\pi_1(G_2, v_2)$. And if $p_{1*}\pi_1(G_1, v_1) \subset p_{2*}\pi_1(G_2, v_2)$, then there is a unique morphism of \mathcal{C} from p_1 to p_2 (see Remark 4.10). This indicates that F is fully faithful. \square

Proposition 4.17. *Let (G, v) be a connected based graph such that $\chi(G) \geq 3$. Then the connected based 2-covering corresponding to $\pi_1(G, v)_{\text{ev}}$ is the second projection $p : (K_2 \times G, (0, v)) \rightarrow (G, v)$.*

Proof. Let $\varphi : L_n \rightarrow G$ be a loop of (G, v) . Since the lift of φ with respect to p is $\tilde{\varphi} : L_n \rightarrow K_2 \times G$ defined by $\tilde{\varphi}(i) = (i \bmod 2, \varphi(i))$. Therefore the terminal point of $\tilde{\varphi}$ is $(0, v)$ if and only if φ is even. Hence $p_*\pi_1(K_2 \times G) = \pi_1(G)_{\text{ev}}$. \square

Let $f : (G, v) \rightarrow (H, w)$ be a based graph map and $p : (K, u) \rightarrow (H, w)$ be a based 2-covering map. In this case, we consider the basepoint of f^*K is (v, u) . We remark that the connectedness of K does not imply the connectedness of f^*K . We write q for the projection $f^*K \rightarrow G$. Then

Proposition 4.18. $q_*(\pi_1(f^*K)) = f_*^{-1}(p_*(\pi_1(K)))$.

Proof. Let \bar{f} denote the second projection $f^*G \rightarrow G$, $(x, y) \mapsto y$. Since $p\bar{f} = fq$, we have $q_*(\pi_2(f^*G)) \subset f_*^{-1}(p_*\pi_1(K))$. Let $[\varphi] \in f_*^{-1}(p_*\pi_1(K))$ and ψ denote the lift of $f\varphi$ with respect to p . Since $[f\varphi] \in p_*\pi_1(K)$, we have ψ is a loop of (K, u) . Then $x \mapsto (\varphi(x), \psi(x))$ is a loop of f^*K and hence $q_*(\pi_1(f^*G)) = f_*^{-1}(p_*(\pi_1(K)))$. \square

5. COMPUTATIONS

In this section, we compute 2-fundamental groups of some graphs including C_n and K_n . Then we obtain a condition for a graph whose chromatic number is 3. For the application of it, we investigate involutions on bipartite graphs. Next we construct some CW-complex whose fundamental group is equal to the 2-fundamental group, and prove van Kampen theorem for 2-fundamental groups.

5.1. Computations of 2-fundamental groups of graphs. Here we compute 2-fundamental groups of C_n and K_n . First we remark that $\pi_1(K_2)$ is trivial by definitions, and hence $\pi_1(C_4)$ is trivial since C_4 is \times -homotopy equivalent to K_2 .

Next we consider the case of K_n for $n \geq 4$.

Theorem 5.1. *For $n \geq 4$, $\pi_1(K_n)_{\text{ev}}$ is trivial. Hence $\pi_1(K_n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 4$.*

Proof. Let $0 \in K_n$ be a base point of K_n . Let $\varphi : L_m \rightarrow G$ be a loop of K_n . Suppose $m \geq 4$. Let $a \in V(K_n)$ be a vertex such that a is not $0 = \varphi(0)$, $\varphi(2)$ or $\varphi(4)$. We define a loop $\psi : L_m \rightarrow K_n$ by $\psi(i) = \varphi(i)$ for $i \neq 1, 3$ and $\psi(1) = \psi(3) = a$. We define a loop $\gamma : L_{m-2} \rightarrow K_n$ by $\gamma(1) = a$ and $\gamma(i) = \varphi(i+2)$ for $i \geq 2$. Then we have $\varphi \simeq \psi \simeq \gamma$. Therefore for a loop $\varphi : L_m \rightarrow K_n$ with $m \geq 4$ is homotopic to a loop with length $m-2$. Since a loop whose length is 2 is 2-homotopic to a trivial path, we have $\pi_1(K_n)_{\text{ev}}$ is trivial. \square

Corollary 5.2. *For $n \geq 4$, connected 2-coverings over K_n are only K_n and $K_2 \times K_n$.*

Next we consider the case of C_n for $n \geq 5$ or $n = 3$.

Theorem 5.3. *$\pi_1(C_r) \cong \mathbb{Z}$ for $r = 3$ or $r \geq 5$. Therefore $\pi_1(K_3) \cong \mathbb{Z}$.*

Proof. First we prove that $\pi_1(L, 0)$ is trivial. Since L is bipartite, it is sufficient to prove that $\pi_1(L, 0)_{\text{ev}}$ is trivial. Let $\varphi : L_{2n} \rightarrow L$ be a loop of $(L, 0)$. Suppose $n \geq 1$. Then $M = \max\{|\varphi(x)| \mid x \in \{0, 1, \dots, n\}\}$ is positive. Let $x_0 \in \{0, 1, \dots, n\}$ with $|\varphi(x_0)| = M$. Then $x_0 \neq 0, 2n$ since $M \neq 0$, and we have $\varphi(x_0 - 1) = \varphi(x_0 + 1)$. Therefore φ is homotopic to a loop with length $2n - 2$. Hence $\pi_1(G, v)_{\text{ev}}$ is trivial.

Therefore $(L, 0) \rightarrow (C_r, 0)$ is a universal 2-covering over $(C_r, 0)$ for $r \geq 5$ or $r = 3$. For a nonnegative integer n , we let $\varphi_n : L_{rn} \rightarrow C_r$ be a loop of $(C_r, 0)$ defined by $\varphi_n(i) = i \pmod{r}$. Then the lift of φ_n with respect to $(L, 0) \rightarrow (C_r, 0)$ is $\tilde{\varphi}_n : L_{rn} \rightarrow L$ defined by $\tilde{\varphi}_n(i) = i$. For a negative integer n , we let φ_n be a loop of $(C_r, 0)$ defined by $\varphi_n(i) = -i \pmod{r}$, and the lift of φ_n with respect to $(L, 0) \rightarrow (C_r, 0)$ is a loop $\tilde{\varphi}_n : L_{|rn|} \rightarrow L$ where $\tilde{\varphi}_n(i) = -i$. From Corollary 4.12, $\pi_1(C_r, 0) = \{[\varphi_n] \mid n \in \mathbb{Z}\}$ and $[\varphi_n] \neq [\varphi_m]$ for $n \neq m$. Since $[\varphi_n] \cdot [\varphi_m] = [\varphi_{n+m}]$ for $n, m \in \mathbb{Z}$, we have $\mathbb{Z} \cong \pi_1(C_r, v)$. \square

Corollary 5.4. *Let G be a graph. If $\chi(G) = 3$, $H_1(G)$ has \mathbb{Z} as a direct summand.*

Proof. We can assume that G is connected. Suppose $\chi(G) = 3$ and $H_1(G)$ does not have \mathbb{Z} as a direct summand. Then there exists a graph homomorphism $f : G \rightarrow K_3$. Since the hypothesis means that there is no surjective group homomorphism from $\pi_1(G)$ to \mathbb{Z} , the induced map $\pi_1(f) : \pi_1(G) \rightarrow \pi_1(K_3) \cong \mathbb{Z}$ is trivial. So there exists a lift $G \rightarrow L$, and we have $\chi(G) \leq \chi(L) = 2$. This contradicts $\chi(G) = 3$. \square

5.2. Involutions of connected bipartite graphs. For an application of Corollary 5.5, we consider involutions of bipartite graphs.

For a graph G , a graph homomorphism $f : G \rightarrow G$ is called an *involution* of G if $f^2 = \text{id}$.

Let G be a connected bipartite graph and $f : G \rightarrow G$ a graph homomorphism. We can easily show that it is independent of the choice of $v \in V(G)$ and a path φ from v to $f(v)$ whether the length of φ is even or odd. We say that a graph homomorphism f is *even* if the length of a path from v to $f(v)$ is even for $v \in V(G)$. We say that f is *odd* if f is not even.

Remark 5.5. Let G be a connected bipartite graph. Then there exist $A_0, A_1 \subset V(G)$ such that

- (i) A_0 and A_1 are independent in G .
- (ii) $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1 = V(G)$.

This unordered pair $\{A_0, A_1\}$ is unique. Let $f : G \rightarrow G$ be a graph homomorphism. Then we have

$$f \text{ is even.} \Leftrightarrow f(A_0) \subset A_0. \Leftrightarrow f(A_1) \subset A_1.$$

and

$$f \text{ is odd.} \Leftrightarrow f(A_0) \subset A_1. \Leftrightarrow f(A_1) \subset A_0.$$

Therefore, if $\tau : G \rightarrow G$ is an odd involution for a connected bipartite graph G , the action of $\mathbb{Z}/2\mathbb{Z}$ induced by τ is a 2-covering action. Indeed, for $v \in A_0$, $N_2(v) \subset A_0$ and $N_2(\tau v) \subset A_1$, hence $N_2(v) \cap N_2(\tau v) = \emptyset$. The case $v \in A_1$ is similar.

Theorem 5.6. *Let G be a connected bipartite graph. Suppose $H_1(G)$ does not have \mathbb{Z} as a direct summand. Let τ be an involution of G . Then we have;*

- (1) *if τ is even, for any 3-coloring $f : G \rightarrow K_3$, there exists $v \in V(G)$ such that $f(v) = f(\tau v)$.*
- (2) *if τ is odd, for any 3-coloring $f : G \rightarrow K_3$, there exists $v \in V(G)$ such that $f(v) \neq f(\tau v)$.*

Proof. Let $v \in V(G)$. We consider v is a basepoint of G .

(1) : Let τ be an even involution of G . We define a graph G_τ by $V(G_\tau) = V(G)$ and $E(G_\tau) = E(G) \cup \{(x, \tau x) \mid x \in V(G)\}$. It is sufficient to prove that $\chi(G_\tau) \geq 4$. Since τ is even, there exists a path $\varphi : L_{2r} \rightarrow G$ such that $\varphi(0) = v$ and $\varphi(2r) = \tau v$. Then we have a loop $\varphi' : L_{2r+1} \rightarrow G_\tau$ defined by $x \mapsto \varphi(x)$ for $x \leq 2r$ and $\varphi(2r+1) = v$. Therefore $\chi(G) \geq 3$.

Let $\varphi : L_n \rightarrow G_\tau$ be a loop of (G_τ, v) . Let $n(\varphi)$ denote $\#\{x \in \{0, 1, \dots, n-1\} \mid (\varphi(x), \varphi(x+1)) \notin E(G)\}$. We show if $n(\varphi) \geq 2$, φ is homotopic to ψ such that $n(\psi) = n(\varphi) - 2$. Let x_φ be the minimum of $\{x \in \{0, 1, \dots, n-1\} \mid (\varphi(x), \varphi(x+1)) \notin E(G)\}$. Since $n(\varphi) \geq 2$, $x_\varphi \neq n-1$. If $(\varphi(x_\varphi+1), \varphi(x_\varphi+2)) \in E(G)$, φ is homotopic to φ' defined by $\varphi'(i) = \varphi(i)$ for $i \neq x+1$ and $\varphi'(x+1) = \tau\varphi(x+1)$. Then $x_{\varphi'} = x_\varphi + 1$ and $n(\varphi) = n(\varphi')$. Therefore φ is homotopic to a loop ψ such that $(\psi(x_\psi+1), \psi(x_\psi+2)) \notin E(G)$ and $n(\psi) = n(\varphi)$. Since $\psi(x_\psi+2) = \tau\psi(x_\psi+1) = \tau^2\psi(x_\psi) = \psi(x_\psi)$, ψ is homotopic to the loop η defined by $\eta(i) = \psi(i)$ for $i \leq x_\psi$ and $\eta(i) = \psi(i+2)$ for $i \geq x_\psi$. Then we have $n(\eta) = n(\varphi) - 2$.

Therefore if $n(\varphi)$ is even, φ is homotopic to a loop of (G, v) . Since $n(\varphi \cdot \varphi) = n(\varphi) + n(\varphi)$, $\alpha^2 \in i_*\pi_1(G, v)$ for every $\alpha \in \pi_1(G_\tau, v)$.

Suppose that there exists a surjective group homomorphism $\Phi : \pi_1(G_\tau, v) \rightarrow \mathbb{Z}$. Let $\alpha \in \Phi^{-1}(1)$. Since $\alpha^2 \in i_*\pi_1(G, v)$, the composition

$$\pi_1(G, v) \xrightarrow{i_*} \pi_1(G_\tau, v) \xrightarrow{\Phi} \mathbb{Z}$$

is not trivial. This contradicts to the assumption of $\pi_1(G, v)$.

(2) : We write H for the quotient graph $G/(\mathbb{Z}/2\mathbb{Z})$, and w be the image of v of the quotient map $p : G \rightarrow H$. We want to show that $\chi(H) \geq 4$. Since τ is odd, there is a path $\varphi : L_{2r+1} \rightarrow G$ such that $\varphi(0) = v$ and $\varphi(2r+1) = \tau v$. Then $p\varphi$ is a loop of (H, w) with odd length, and we have $\chi(H) \geq 3$.

Since τ is odd, the quotient map p is a connected based double 2-covering over (H, w) . Therefore $p_*\pi_1(G, v)$ is a subgroup of $\pi_1(H, w)$ whose index is 2 (see Lemma 4.11.). Hence there are no surjective group homomorphisms from $\pi_1(H, w)$ to \mathbb{Z} . Therefore we have $\chi(H) \neq 3$ from Corollary 5.4. \square

Remark 5.7. (1) We can see that the chromatic number of G_τ in the proof of (1) of previous theorem is 4 if τ has no fixed point. In fact, let 2-coloring $f : G \rightarrow K_2$ and $A \subset V(G)$ such that $A \cap \tau A = \emptyset$ and $A \cup \tau A = V(G)$. Then we can obtain a graph homomorphism $g : V(G_\tau) \rightarrow K_4$ defined by $g(x) = 2f(x)$ for $x \in A$ and $g(x) = 2f(x) + 1$ for $x \in \tau A$.

(2) The chromatic number of H in the proof of (2) of previous theorem is not determined. In fact, If G is $K_2 \times K_n$ and τ is defined by $\tau(\epsilon, x) = (\epsilon + 1 \bmod 2, x)$, then H is K_n .

Example 5.8. Let r be a positive integer. Let n_1, n_2, \dots, n_r be positive integers and s a integer with $0 \leq s \leq r - 2$. We define the graph $G(n_1, \dots, n_r; s)$ by

$$V(G(n_1, \dots, n_r; s)) = \{(x_1, \dots, x_r) \mid x_i \in \{0, 1, \dots, n_i\} \text{ and } \#\{i \mid x_i = 0 \text{ or } n_i\} \geq s\}$$

$$E(G(n_1, \dots, n_r; s)) = \{((x_1, \dots, x_r), (y_1, \dots, y_r)) \mid \sum_{i=1}^n |x_i - y_i| = 1\}.$$

$G(n_1, \dots, n_r; s)$ is bipartite since there exists a graph map $f : G(n_1, \dots, n_r; s) \rightarrow K_2$ defined by $f(x_1, \dots, x_r) = x_1 + \dots + x_r \pmod{2}$. We will prove that $\pi_1(G(n_1, \dots, n_r; s))$ is trivial in Example 5.15. G has an involution τ defined by $\tau(x_1, \dots, x_r) \mapsto (n_1 - x_1, \dots, n_r - x_r)$. τ is even if $N = n_1 + \dots + n_r$ is even and is odd if N is odd. Therefore, for each 3-coloring $f : G(n_1, \dots, n_r; s) \rightarrow K_3$, there is $x \in V(G(n_1, \dots, n_r; s))$ such that $f(\tau x) = f(x)$ ($f(\tau x) \neq f(x)$) if N is even (odd, respectively).

Example 5.9. (1): A 3-coloring f of C_8 such that there exists no $v \in C_8$ such that $f(\tau v) = f(v)$, where τ is an antipodal map of C_8 . We remark τ is even.

(2): A 3-coloring g of C_6 such that $g\tau = g$ where τ is an antipodal map of C_6 . We remark τ is odd.

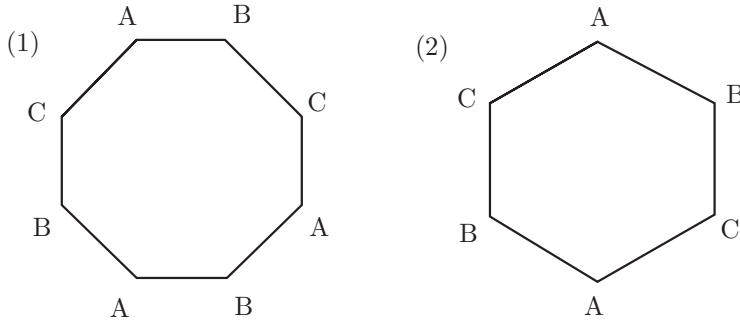


Figure 1.

5.3. Van Kampen's theorem for 2-fundamental groups. First, for a given graph G , we construct a 2-dimensional CW complex $|G|$ whose fundamental group is isomorphic to $\pi_1(G)$. In the construction of $|G|$ we need the following definitions.

Definition 5.10. Let G be a graph. A graph map $C_4 \rightarrow G$ is called a *square* of G . A square $\sigma : C_4 \rightarrow G$ of G is said to be *degenerate* if $\sigma(0) = \sigma(2)$ or $\sigma(1) = \sigma(3)$. A square σ of G is said to be *nondegenerate* if σ is not degenerate.

Let σ, σ' be squares of G . We say that σ is *equivalent* to σ' if there is $k \in \{0, 1, 2, 3\}$ such that $\sigma(i) = \sigma'(i + k)$ or $\sigma(i) = \sigma'(k - i)$ for each $i \in V(C_4)$.

Example 5.11. The examples of nondegenerate squares

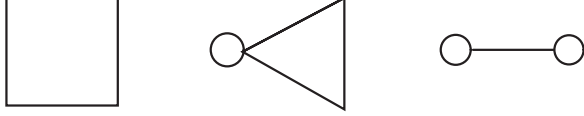


Figure 2

We begin to construct $|G|$. First the set of 0-cells of $|G|$ is $V(G)$.

We regard $E(G)$ as a $(\mathbb{Z}/2\mathbb{Z})$ -set by $\tau(x, y) = (y, x)$ where τ is a generator of $\mathbb{Z}/2\mathbb{Z}$. For each $\{(x, y), (y, x)\} \in E(G)/(\mathbb{Z}/2\mathbb{Z})$, we attach a 1-cell whose end points are x and y . We fix an orientation of a loop of each looped vertex.

2-cells of $|G|$ are attached as follows. For each looped vertex, we attach a 2-cell twice along given orientation of the loop. For each equivalence class α of a nondegenerate square, we let some representative $\sigma_\alpha \in \alpha$ and attached a cell along the loop

$$\sigma(0) \rightarrow \sigma(1) \rightarrow \sigma(2) \rightarrow \sigma(3).$$

This is the construction of $|G|$.

Theorem 5.12. $\pi_1(G, v) \cong \pi_1(|G|, v)$ for a based graph (G, v) . $H_1(G) \cong H_1(|G|; \mathbb{Z})$ for a graph G .

For the proof of Theorem 5.12, we need the next lemma. We write I for the topological space $[0, 1]$.

Lemma 5.13. Let X be a 1-dimensional CW complex, $v, w \in X^0$. Let α be a homotopy class of paths of X from v to w which is not homotopic to a constant path. Then there exists a unique representative φ_α of α satisfying following properties.

- (1) There exists a positive integer N_α such that $\varphi_\alpha^{-1}(X^0) = \{\frac{i}{N_\alpha} \mid i \in \{0, 1, \dots, N_\alpha\}\}$.
- (2) For each $i \in \{0, 1, \dots, N_\alpha - 1\}$, a path $I \rightarrow X$, $t \mapsto \varphi_\alpha(\frac{i+t}{N_\alpha})$ or $t \mapsto \varphi_\alpha(\frac{i+1-t}{N_\alpha})$ is a characteristic map of 1-cell whose endpoints are $\varphi_\alpha(\frac{i}{N_\alpha})$ and $\varphi_\alpha(\frac{i+1}{N_\alpha})$.
- (3) There is no $i \in \{1, \dots, N_\alpha - 1\}$ such that $\varphi_\alpha(\frac{i-t}{N_\alpha}) = \frac{i+t}{N_\alpha}$ for all $t \in I$.

Definition 5.14. Let X be a 1-dimensional CW complex, $v, w \in X^0$. Let α be the homotopy class of paths of X from v to w and $\varphi \in \alpha$. Then φ is called a *canonical representative* of α if φ is constant or $\varphi = \varphi_\alpha$ where φ_α is in Lemma 5.13.

We begin to prove Lemma 5.13.

Proof. We can assume X is simply-connected.

Let $\varphi \in \alpha$. Let $x_0 = 0$, $v_0 = v$, $x_1 = \inf(\varphi^{-1}(X^0 - \{v_0\}))$ and $v_1 = \varphi(x_1)$. We remark that since $\varphi^{-1}(X^0 - \{v_0\})$ is closed in I , $v_0 \neq v_1 \in X^0$. If $\varphi^{-1}(X^0 - \{v_1\}) \cap [x_1, 1] \neq \emptyset$, set $x_2 = \inf(\varphi^{-1}(X^0 - \{v_1\}) \cap [x_1, 1])$ and $v_2 = \varphi(x_2)$. After finite this operations, we have x_n such that $\varphi^{-1}(X^0 - \{v_n\}) \cap [x_n, 1] = \emptyset$.

In fact, if not, there exists an infinite sequence $(x_i)_i$ of I such that $x_i < x_{i+1}$, $\varphi(x_i) \in X^0$ and $\varphi(x_i) \neq \varphi(x_{i+1})$. Set $x_\infty = \sup\{x_i \mid i \in \mathbb{N}\}$. Since $\varphi^{-1}(X^0)$ is closed in I , $\varphi(x_\infty) \in X^0$. Since $\varphi^{-1}(x_\infty)$ is open in $\varphi^{-1}(X^0)$, there exists $n \in \mathbb{N}$ such that for all $m \geq n$, we have $\varphi(x_m) = \varphi(x_\infty)$. But this contradicts to $\varphi(x_n) \neq \varphi(x_{n+1})$.

Therefore we have a finite sequence $v = x_0, \dots, x_n = w$. Then φ is homotopic to $\psi : I \rightarrow X$ satisfying $\psi^{-1}(X^0) = \{\frac{i}{n} \mid i \in \{0, 1, \dots, n\}\}$ and $\psi(\frac{i}{n}) = x_i$ and (2).

If ψ does not satisfy (3), there exists $i \in \{1, \dots, N-1\}$ such that $\psi(i+t) = \psi(i-t)$ for $t \in I$. We define a path $\psi' : I \rightarrow X$ by setting $\psi'(\frac{j+t}{N-2}) = \psi(\frac{j+t}{N})$ for $t \in I$, $j < i$ and $\psi'(\frac{j+t}{N-2}) = \psi(\frac{j+2+t}{N})$ for $t \in I$, $j \geq i$. Then ψ' is obviously homotopic to ψ . After finite these operations, we have φ_α satisfying (1), (2), (3).

We prove the uniqueness of φ_α . Suppose φ_α is not unique. Then we can easily see that there exists a loop $\varphi : I \rightarrow X$ satisfying (1), (2), (3). If there exists $\varphi(\frac{i}{N}) = \varphi(\frac{j}{N})$ for $i < j$, let φ' be a loop defined by the composition

$$[0, 1] \longrightarrow [\frac{i}{N}, \frac{j}{N}] \xrightarrow{\varphi} X.$$

Therefore we can assume $\varphi(\frac{i}{N}) \neq \varphi(\frac{j}{N})$ for $i < j$.

Let Y be a subcomplex $X \setminus \varphi((0, \frac{1}{N}))$. Then the composition

$$S^1 \approx [0, 1]/\{0, 1\} \xrightarrow{\varphi} X \longrightarrow X/Y \approx S^1$$

has degree 1 or -1 . This is a contradiction since $\pi_1(X)$ is trivial. \square

We begin to prove Theorem 5.12.

Proof. Let $\varphi : L_n \rightarrow G$ be a loop of (G, v) . We write $\Phi(\varphi)$ for the loop of $(|G|, v)$ such that $\varphi(0) \rightarrow \varphi(1) \rightarrow \dots \rightarrow \varphi(n)$. Then Φ induces a group homomorphism $\overline{\Phi} : \pi_1(G, v) \rightarrow \pi_1(|G|, v)$. We define the group homomorphism $\Psi : \pi_1(|G|^1, v) \rightarrow \pi_1(G, v)$ as follows. We set $\Psi(1) = 1$. For $1 \neq \alpha \in \pi_1(|G|^1, v)$, let φ_α be the canonical representation of α . Then we define the loop $\Psi(\alpha) : L_{N_\alpha} \rightarrow G$ of (G, v) by setting $\Psi(\alpha)(i) = \varphi_\alpha(\frac{i}{N_\alpha})$. Then Ψ induces $\overline{\Psi} : \pi_1(|G|, v) \rightarrow \pi_1(G, v)$ and $\overline{\Psi}$ is the inverse of $\overline{\Phi}$. \square

Example 5.15. Let $G = G(n_1, \dots, n_r; s)$ ($s \leq r-2$) in Example 5.8. From Theorem 5.13, $\pi_1(G)$ is isomorphic to the fundamental group of a topological space $Y_{r,s}$ where

$$Y_{r,s} = \{(x_1, \dots, x_r) \in [0, 1]^n \mid \#\{i \mid x_i = 0 \text{ or } 1\} \geq s\}.$$

First we remark that $Y_{r,0}$ is simply connected for $r \geq 2$, and $Y_{r,s}$ for $s \leq r-1$ is connected. Let (r, s) for $r \geq s$,

$$Y_{r,s}^+ = \{(x_1, \dots, x_r) \in [0, 1]^n \mid \#\{i \mid x_i = 0 \text{ or } 1\} \geq s, x_r < 1\}$$

$$Y_{r,s}^- = \{(x_1, \dots, x_r) \in [0, 1]^n \mid \#\{i \mid x_i = 0 \text{ or } 1\} \geq s, x_r > 0\}$$

Since $Y_{r-1,s-1}$ is the deformation retract of $Y_{r,s}^+$ and $Y_{r,s}^-$ and $Y_{r,s}^+ \cap Y_{r,s}^-$ is connected since $Y_{r-1,s}$ is the deformation retract of $Y_{r,s}^+ \cap Y_{r,s}^-$. Hence $Y_{r,s}$ for $s \leq r-2$ is simply connected from van Kampen's theorem.

Before giving the statement of van Kampen's theorem for 2-fundamental groups, we give some definitions about squares.

Definition 5.16. Let G be a graph and $\sigma, \sigma_1, \sigma_2$ be squares of G . We write $\sigma_1 \cup \sigma_2 = \sigma$ or $\sigma_2 \cup \sigma_1 = \sigma$ if one of the following two conditions are satisfied.

- (1) $\sigma(i) = \sigma_1(i)$ for $i \neq 2$ and $\sigma(i) = \sigma_2(i)$ for $i \neq 0$ and $\sigma_1(2) = \sigma_2(0)$.
- (2) $\sigma(i) = \sigma_1(i)$ for $i \neq 3$ and $\sigma(i) = \sigma_2(i)$ for $i \neq 1$ and $\sigma_1(3) = \sigma_2(1)$.

Let G be a graph and σ be a square of G . We define a decomposition sequence of σ as follows :

- (1) (σ) is a decomposition sequence.
- (2) If $(\sigma_1, \dots, \sigma_n)$ be a decomposition sequence of σ and $\sigma_i = \tau \cup \tau'$, then $(\sigma_1, \dots, \sigma_{i-1}, \tau, \tau', \sigma_{i+1}, \dots, \sigma_n)$ is a decomposition sequence of σ .

We say that σ is decomposed into $\sigma_1, \sigma_2, \dots, \sigma_n$ if $(\sigma_1, \dots, \sigma_n)$ is a decomposition sequence of σ .

Let (G, v) be a based graph. Let $\{G_\alpha\}_{\alpha \in A}$ be a family of subgraphs of G such that $v \in V(G_\alpha)$ for each $\alpha \in A$. We write i_α for the inclusion $G_\alpha \rightarrow G$ and $j_{\alpha\beta}$ for the inclusion $G_\alpha \cap G_\beta \rightarrow G_\alpha$. We define the group homomorphism $\Phi : *_{\alpha \in A} \pi_1(G_\alpha, v) \rightarrow \pi_1(G, v)$ by the free product of all $\pi_1(i_\alpha)$. Let N be the normal subgroup generated by $\{j_{\alpha\beta}^{-1}(x)j_{\beta\alpha}(x) \mid \alpha, \beta \in A, x \in \pi_1(G_\alpha \cap G_\beta, v)\}$. Since $i_\alpha j_{\alpha\beta} = i_\beta j_{\beta\alpha}$, N is in the kernel of Φ . Therefore Φ induces the group homomorphism $\tilde{\Phi} : *_{\alpha \in A} \pi_1(G_\alpha, v)/N \rightarrow \pi_1(G, v)$. The statement of the following theorem is that $\tilde{\Phi}$ is isomorphism under some conditions. This is an analogy of van Kampen theorem.

Theorem 5.17. *Let (G, v) be a graph. Let $\{G_\alpha\}_{\alpha \in A}$ be a family of subgraphs of G satisfying following properties.*

- (i) $\bigcup_{\alpha \in A} G_\alpha = G$ and $v \in V(G_\alpha)$ for all $\alpha \in A$.
- (ii) $G_\alpha \cap G_\beta \cap G_\gamma$ is connected for all $\alpha, \beta, \gamma \in A$.
- (iii) Every nondegenerate square of G is decomposed into squares contained in some G_α .

*Then the homomorphism $\tilde{\Phi} : *_{\alpha \in A} \pi_1(G_\alpha, v)/N \rightarrow \pi_1(G, v)$ defined above is an isomorphism.*

Proof. In this case, $\pi_1(|G|, v) \cong \pi_1(\bigcup_\alpha |G_\alpha|, v)$ and $|G_\alpha \cap G_\beta| = |G_\alpha| \cap |G_\beta|$. Therefore we can deduce this theorem from van Kampen theorem of topology. \square

Example 5.18. Let $\{(G_\alpha, v_\alpha)\}_\alpha$ be a family of based graphs such that v_α is not looped for all α . In this case, a nondegenerate square of $\bigvee_\alpha G_\alpha$ is contained in some G_α . Therefore we have $\pi_1(\bigvee_\alpha G_\alpha) \cong *_{\alpha} \pi_1(G_\alpha)$.

Let G be a graph. We write $H_0(G)$ for the free abelian group generated by the set of all connected components of G .

Theorem 5.19. *Let G be a graph and K_1, K_2 subgraphs of G . If every nondegenerate square of G is decomposed in squares of K_1 or K_2 , there exists an exact sequence*

$$H_1(K_1 \cap K_2) \rightarrow H_1(K_1) \oplus H_1(K_2) \rightarrow H_1(K_1 \cup K_2) \rightarrow H_0(K_1 \cap K_2) \rightarrow H_0(K_1) \oplus H_0(K_2) \rightarrow H_0(K_1 \cup K_2) \rightarrow 0$$

Proof. In this case, $H_1(|K_1 \cup K_2|; \mathbb{Z}) = H_1(|K_1| \cup |K_2|; \mathbb{Z})$. Therefore we can deduce this theorem from Mayer-Vietoris sequence of $H_*(-; \mathbb{Z})$. \square

6. FUNDAMENTAL GROUPS OF NEIGHBORHOOD COMPLEXES

In this section, we prove the following theorem.

Theorem 6.1. *Let (G, v) be a based graph where v is not isolated. Then we have a group isomorphism*

$$\pi_1(G, v)_{\text{ev}} \cong \pi_1(|\mathcal{N}(G)|, v).$$

and this isomorphism is natural with respect to based graph homomorphisms.

Corollary 6.2. *Let G be a graph. If $\chi(G) = 3$, $H_1(\mathcal{N}(G); \mathbb{Z})$ has \mathbb{Z} as a direct summand.*

Proof. This is deduced from Theorem 6.1 and Corollary 5.4. \square

Remark 6.3. For a locally finite graph G , the map $|\text{Hom}(K_2, G)| \rightarrow |\mathcal{N}(G)|$ induced by the poset map $\eta \mapsto \eta(0)$ is a homotopy equivalence, see [1]. Therefore we have similar statements of Theorem 6.1 and Corollary 6.2 for $\text{Hom}(K_2, G)$. But since $\text{Hom}(K_2, G)$ has no canonical point for a based graph (G, v) , and the statement and the proof of Theorem 6.1 become more complicated.

Corollary 6.4. *Let (G, v) and (H, w) be based graphs where v and w are not isolated. Then*

$$(p_{1*}, p_{2*}) : \pi_1(G \times H, (v, w))_{\text{ev}} \rightarrow \pi_1(G, v)_{\text{ev}} \times \pi_1(H, w)_{\text{ev}}, \alpha \mapsto (p_{1*}\alpha, p_{2*}\alpha)$$

is an isomorphism where $p_1 : G \times H \rightarrow G$ and $p_2 : G \times H \rightarrow H$ are projections.

Proof. Then this is deduced from the fact that $(|p_{1*}|, |p_{2*}|) : |\mathcal{N}(G \times H)| \rightarrow |\mathcal{N}(G)| \times |\mathcal{N}(H)|$ is a homotopy equivalence. We think this is well-known, but we give the proof for self-contained.

We write $FN(G)$ for the face poset of $\mathcal{N}(G)$. Then we have the two poset maps

$$p : FN(G \times H) \rightarrow FN(G) \times FN(H), \sigma \mapsto (p_1(\sigma), p_2(\sigma))$$

$$i : FN(G) \times FN(H) \rightarrow FN(G \times H), (\sigma, \tau) \mapsto \sigma \times \tau.$$

It is easy to show that p and i are well-defined poset maps, $pi = \text{id}$ and $ip(\sigma) \geq \sigma$ for $\sigma \in FN(G \times H)$. Hence ip is an ascending closure map and $|p| \cong |(p_{1*}, p_{2*})|$ is a homotopy equivalence. \square

To prove Theorem 6.1, we first construct the group homomorphism $\overline{\Phi} : \pi_1(G, v) \rightarrow \pi_1(\mathcal{N}(G), v)$.

Let φ be a loop with length $2n$. We remark that $\varphi(2i-2), \varphi(2i) \in N(\varphi(2i-1))$, so we have $\{\varphi(2i-2), \varphi(2i)\}$ is a simplex of $\mathcal{N}(G)$. For a vertex $a, b \in \mathcal{N}(G)$ such that $\{a, b\}$ is a simplex of $\mathcal{N}(G)$, let $\gamma_{ba} : I \rightarrow |\mathcal{N}(G)|$ denote a path $\gamma_{ba}(t) = (1-t)a + tb$. Let (a_0, \dots, a_m) be a finite sequence of vertices of $\mathcal{N}(G)$ such that $\{a_{i-1}, a_i\} \subset \mathcal{N}(G)$ for $1 \leq i \leq m$. Then we write

$$a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_m$$

for a path $t \mapsto \gamma_{a_i a_{i-1}}(mt - i)$ for $\frac{i-1}{m} \leq t \leq \frac{i}{m}$. In this notation, the loop $\Phi(\varphi)$ is defined by

$$\varphi(0) \rightarrow \varphi(2) \rightarrow \dots \rightarrow \varphi(2n).$$

We prove Φ induces a group homomorphism $\overline{\Phi} : \pi_1(G, v) \rightarrow \pi_1(|\mathcal{N}(G)|, v)$.

Let φ, ψ be loops of (G, v) with even lengths such that $\varphi \simeq \psi$. We want to say $\Phi(\varphi) \simeq \Phi(\psi)$. It is sufficient to show that (φ, ψ) satisfies the condition (i) or (ii)' in the definition of 2-homotopy of paths. Suppose (φ, ψ) satisfies (i). Namely $l(\varphi) = 2n$ then $l(\psi) = 2n + 2$ and there exists $x \in \{1, 2, \dots, 2n-1\}$ such that $\varphi(i) = \psi(i)$ for $(i \leq x)$ and $\varphi(i) = \psi(i+2)$ for $i \geq x$. If x is even, $\Phi(\varphi) \simeq \Phi(\psi)$ is obvious since $\Phi(\varphi)$ is the path

$$v = \varphi(0) \rightarrow \varphi(2) \rightarrow \dots \rightarrow \varphi(x-2) \rightarrow \varphi(x) \rightarrow \varphi(x+2) \rightarrow \dots \rightarrow \varphi(2n) = v$$

and $\Phi(\psi)$ is the loop

$$v = \varphi(0) \rightarrow \varphi(2) \rightarrow \dots \rightarrow \varphi(x-2) \rightarrow \varphi(x) \rightarrow \varphi(x) \rightarrow \varphi(x+2) \rightarrow \dots \rightarrow \varphi(2n) = v.$$

Suppose x is odd. We remark that $\psi(x-1)(=\varphi(x-1))$, $\psi(x+1)$ and $\psi(x+3)(=\varphi(x+1))$ is in the neighborhood of $\psi(x)(=\varphi(x)=\varphi(x+2))$. Therefore $\Phi(\varphi)$ is homotopic to the loop

$$\varphi(0) \rightarrow \varphi(2) \rightarrow \cdots \rightarrow \varphi(x-1) \rightarrow \psi(x+1) \rightarrow \varphi(x-1) \rightarrow \varphi(x+1) \rightarrow \cdots \rightarrow \varphi(2n)$$

and this is homotopic to the loop

$$\varphi(0) \rightarrow \varphi(2) \rightarrow \varphi(x-1) \rightarrow \psi(x+1) \rightarrow \varphi(x+1) \rightarrow \cdots \rightarrow \varphi(2n).$$

Since the last one is equal to $\Phi(\psi)$, we have $\Phi(\varphi) \simeq \Phi(\psi)$.

Suppose (φ, ψ) satisfies the condition (ii)'. Namely $l(\varphi) = l(\psi) = 2n$ and there is $x \in \{1, 2, \dots, 2n-1\}$ such that $\varphi(i) = \psi(i)$ for $i \neq x$. If x is odd, then $\Phi(\varphi) = \Phi(\psi)$. Suppose x is even. We remark that $\{\varphi(x-2), \varphi(x), \psi(x)\} \subset N(\psi(x-1))$ and $\{\varphi(x), \psi(x), \varphi(x+2)\} \subset N(\varphi(x+1))$. Therefore

$$\varphi(0) \rightarrow \varphi(2) \rightarrow \cdots \rightarrow \varphi(x-2) \rightarrow \varphi(x) \rightarrow \varphi(x+2) \varphi \cdots \varphi(2n)$$

is homotopic to

$$\varphi(0) \rightarrow \varphi(2) \rightarrow \cdots \rightarrow \varphi(x-2) \rightarrow \varphi(x) \rightarrow \psi(x) \rightarrow \varphi(x) \rightarrow \varphi(x+2) \rightarrow \cdots \rightarrow \varphi(2n)$$

and this is homotopic to

$$\varphi(0) \rightarrow \varphi(2) \rightarrow \cdots \rightarrow \varphi(x-2) \rightarrow \psi(x) \rightarrow \varphi(x+2) \rightarrow \cdots \rightarrow \varphi(2n).$$

Therefore we have $\Phi(\varphi) \simeq \Phi(\psi)$.

Hence Φ induces $\overline{\Phi} : \pi_1(G, v) \rightarrow \pi_1(|\mathcal{N}(G)|, v)$. From the definition of Φ , $\overline{\Phi}$ is a group homomorphism and is natural with respect to based graph homomorphisms.

Next we construct the inverse of $\overline{\Phi}$. We define the $\Psi : \pi_1(|\mathcal{N}(G)|^1, v) \rightarrow \pi_1(G, v)_{\text{ev}}$ as follows. First we set $\Psi(1) = 1$. Let $\alpha \in \pi_1(|\mathcal{N}(G)|^1, v)$ be a non-identity element and φ_α a canonical representative of α . Then there exists $n \in \mathbb{N}$ such that $\varphi_\alpha^{-1}(|\mathcal{N}(G)|^0) = \{\frac{i}{n} \mid 0 \leq i \leq n\}$. Then we define $\Psi(\alpha)$ by the 2-homotopy class of

$$\varphi(0) \rightarrow v_1 \rightarrow \varphi(\frac{1}{n}) \rightarrow v_2 \rightarrow \cdots \rightarrow v_n \rightarrow \varphi(1)$$

where v_i is a vertex of $V(G)$ such that $\varphi(\frac{i-1}{n}), \varphi(\frac{i}{n}) \in N(v_i)$. From the definition of 2-homotopy of paths, the homotopy class of the above path is independent of the choice of v_i . We can easily see that Ψ is a group homomorphism. We want to show that Ψ induces a group homomorphism $\overline{\Psi} : \pi_1(|\mathcal{N}(G)|, v) \rightarrow \pi_1(G, v)_{\text{ev}}$. Then it is sufficient to show that, for each loop γ of $(|\mathcal{N}(G)|^1, v)$ which is homotopic to an attaching map of a 2-cell of $|\mathcal{N}(G)|$, $\Psi([\gamma])$ is nullhomotopic.

Let $\{y_0, y_1, y_2\}$ be a 2-simplex of $|\mathcal{N}(G)|$. Let γ be the loop of $(|\mathcal{N}(G)|^1, v)$ written by

$$v = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = y_0 \rightarrow y_1 \rightarrow y_2 \rightarrow y_0 = x_n \rightarrow \cdots \rightarrow x_1 \rightarrow x_0.$$

Then $\Psi([\gamma])$ is a homotopy class of a loop $\varphi : L_{4n+6} \rightarrow G$ such that $\varphi(2i) = x_i = \varphi(4n+6-2i)$ for $i = 0, 1, \dots, n$ and $\varphi(2n+2) = y_1$ and $\varphi(2n+4) = y_2$. Since $\{y_0, y_1, y_2\}$ is a simplex of $\mathcal{N}(G)$, there exists $w \in V(G)$ such that $\{y_0, y_1, y_2\} \subset N(w)$. Therefore we can assume $\varphi(2n+1) = \varphi(2n+3) = \varphi(2n+5) = w$. Thus φ is homotopic to $\psi : L_{4n} \rightarrow G$ such that $\psi(2i) = x_i = \psi(4n-2i)$ for $i = 0, 1, \dots, n$. But ψ is obviously nullhomotopic.

Therefore Ψ induces a group homomorphism $\overline{\Psi} : \pi_1(|\mathcal{N}(G)|, v) \rightarrow \pi_1(G, v)_{\text{ev}}$. From the definition, $\overline{\Psi}$ is the inverse of $\overline{\Phi}$. This completes the proof of Theorem 6.1.

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